Recap 18.965

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Contents

1 Intrinsic and extrinsic geometry

A central motivation of differential and Riemannian geometry is to abstract from an ambient space where our geometric objects reside.

1.1 Extrinsic geometry

Definition 1.1. **Extrinsic geometry** involves properties of a surface that depend on how the surface is embedded in the surrounding space.

Example 1.2.

- *In the plane:* The curvature of a circle, or a more general curve on a plane is an example of extrinsic quantity. The Euclidean distance (length of a straight segment) of the plane is an *extrinsic* distance on the curve.
- *In space:* A sphere (or the surface of Earth), viewed as a sphere in 3-dimensional space has an extrinsic distance corresponding to drilling straight holes through our planet.
- *In space:* A cylinder and a plane have different extrinsic geometries (the cylinder is curved in space).
- **Mathematical Tools:**
  - The **second fundamental form** is used to compute extrinsic curvature.
  - The relationship between intrinsic and extrinsic curvature is captured by results such as the **Gauss-Codazzi-Mainardi equations**.

1.2 Intrinsic geometry

Definition 1.3. **Intrinsic geometry** refers to properties of a surface that are determined by the surface itself, independent of how it is embedded in the surrounding space. It actually does not require an ambient space at all.

Two "subspaces" (soon called submanifolds) have the same intrinsic geometry if they are **isometric**, i.e. there is an *isometry* between them.

Definition 1.4. An **isometry** between Σ_1 and Σ_2 is a one-to-one map $\Phi : \Sigma_1 \to \Sigma_2$ preserving *intrinsic* distances: $d_{\Sigma_1}(x, y) = d_{\Sigma_2}(\Phi(x), \Phi(y))$ for all $x, y \in \Sigma_1$.

Examples:

- *In the plane:* The arc-length lets one define the intrinsic distance between points on a curve. Two curves with the same length have the same intrinsic geometry: they are isometric.
- *In space:* On Earth, the **geodesics** (shortest paths) are pieces of great circles, such as the equator or the lines of longitude on Earth.
- *In space:* A cylinder is (locally) isometric to a flat piece of plane: just unroll it!

2 Submanifolds of $\mathbb{R}^n$

Submanifolds of Euclidean spaces (or other manifolds) are the prototypical examples of manifolds. A submanifold of $\mathbb{R}^n$ is a subset that locally looks like a lower-dimensional Euclidean space. There are several equivalent ways to define submanifolds, and each perspective offers different insights.

Example 2.1.

- **Level Sets:** A common way to define a submanifold is as the level set of a smooth function. For example, if $F : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function, then the set

$$N = \{x \in \mathbb{R}^n \mid F(x) = c\}$$

  is a submanifold of $\mathbb{R}^n$, provided that the differential dF_x has full rank at every point $x \in N$. The **implicit function theorem**, guarantees that locally, N can be expressed as the graph of a function.

- **Parametrized Submanifolds:** Another way to define a submanifold is through a smooth parametrization. For example, if $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is a smooth map with injective differential (which imposes $k \leqslant n$), then the image $N = \varphi(U)$ is a submanifold of $\mathbb{R}^n$. This perspective is more concrete: submanifolds are given explicitly in terms of coordinates.

- **Inverse Function Theorem:** If a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ has a non-degenerate Jacobian at a point, then locally near that point, F behaves like a diffeomorphism. This implies that locally, the image of a submanifold under such a map is also a submanifold.

The *differentiability* of F or φ will be crucial to give our (sub)manifolds a differentiable structure.

Note 2.2. See later sections for general definitions of submanifolds.

Example 2.3.

- *Circle in* $\mathbb{R}^2$*:* The unit circle $\mathbb{S}^1$ can be defined as the level set $\{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1\}$ in $\mathbb{R}^2$, or parametrized by $\theta \mapsto (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi)$.

- *Sphere in* $\mathbb{R}^3$*:* The unit sphere $\mathbb{S}^2$ can be defined as the level set $\{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$, or parametrized by spherical coordinates

  - θ: The polar angle, measured from the positive z-axis. It ranges from 0 to π.
  - ϕ: The azimuthal angle, measured in the xy-plane from the positive x-axis. It ranges from 0 to 2π.

  Given the spherical coordinates (θ, ϕ), the corresponding Cartesian coordinates (x, y, z) on the unit sphere are given by:

$$\begin{cases} x = \sin\theta\cos\phi, \\ y = \sin\theta\sin\phi, \\ z = \cos\theta. \end{cases}$$

  - The angle θ determines the "height" of the point on the sphere, with $\theta = 0$ at the north pole and $\theta = \pi$ at the south pole.
  - The angle ϕ determines the "direction" around the z-axis, making a full circle as ϕ goes from 0 to 2π.

- *Cylinder in* $\mathbb{R}^3$*:* The cylinder over the unit circle $\mathbb{R} \times \mathbb{S}^1$ can be defined as the level set $\{y^2 + z^2 = 1\}$ in $\mathbb{R}^3$ (x taking any real value). It can also be parametrized as: $(x, \theta) \mapsto (x, \cos\theta, \sin\theta)$

- 2-*torus in* 4-*space* $\mathbb{T}^2 \subset \mathbb{R}^4$ defined as

$$\mathbb{T}^2 := \left\{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 = z^2 + t^2 = 1\right\}.$$

  Note that this is a submanifold of the 3-sphere of radius $\sqrt{2}$.

Note 2.4. The notations $\mathbb{S}^n$, $\mathbb{R}^n$ etc. will have different meanings along the class (differentiable manifold, submanifold, Riemannian manifold, topological manifold). I will try to redefine them every time I use them as examples.

3 Differentiable manifolds

Differentiable manifolds are a central object of study in differential geometry, providing a framework for generalizing the concepts of calculus to more abstract spaces. The partition of unity is a powerful tool that allows us to work with global constructions on manifolds by piecing together local data.

3.1 Definition of a manifold

Intuition 3.1. A differentiable manifold is composed of many Euclidean-looking charts that agree with each other in a differentiable way.

Definition 3.2. A **manifold** is a Hausdorff topological space that locally resembles Euclidean space and is equipped with an additional structure called an **atlas**.

- **Chart**: A chart on a topological space M is a pair (U, φ), where:
  - U is an open subset of M.
  - $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism, meaning φ is a continuous, bijective map with a continuous inverse. The map φ is called a **coordinate map** or **coordinate chart**, and it provides a coordinate system for points in U.
- **Atlas**: An atlas on M is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that:
  - The open sets $\{U_\alpha\}$ cover M, i.e., $M = \bigcup_{\alpha \in A} U_\alpha$.
  - For any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) in the atlas, if $U_\alpha \cap U_\beta \neq \emptyset$, then the **transition map** $\varphi_\beta \circ \varphi_\alpha^{-1}$ defined on $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n$ is a smooth (infinitely differentiable) map. The local coordinate representations on overlapping charts are *smoothly compatible*.
- **Manifold**: A topological space M equipped with a countable atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is called a n**-dimensional manifold**.

Note 3.3. The charts are also called coordinate system since we may decompose their images as coordinates: $\phi(p) = (x_1(p), \dots, x_n(p))$.

Note 3.4. Two atlas $(U_\alpha, \phi_\alpha)_\alpha$ and $(V_\beta, \psi_\beta)_\beta$ are equivalent if their union is an atlas. That is, if (U_α, ϕ_α) and (V_β, ψ_β) are two charts of the first and second atlas, then the compositions $\phi_\alpha \circ \psi_\beta^{-1}$ are smooth on the open sets where they are defined.

Note 3.5. A topological manifold is locally compact, locally pathconnected, locally contractible, separable, paracompact, metrizable.

Remark 3.6. A theorem of Whitney also shows that C^1-manifolds and C^k for $k \geqslant 1$ or even C^∞-manifolds are all equivalent. The charts only agree in a continuous way on "topological" (or C^0) manifolds–this is a weaker notion. Topological manifolds may be very different from differentiable ones from dimension 3: there are uncountably many non equivalent differentiable structures on $\mathbb{R}^4$ (but only one for all $\mathbb{R}^n$ for $n \neq 4$).

Examples:

- The n-dimensional Euclidean space $\mathbb{R}^n$ itself is a trivial example of a differentiable manifold.
- The n-dimensional sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.
- The torus $\mathbb{T}^2$, which can be viewed as the product of two circles $\mathbb{S}^1 \times \mathbb{S}^1$, is a 2-dimensional differentiable manifold.
- A submanifold of dimension k in a n-dimensional manifold is a k-dimensional manifold.

3.2 Smooth maps and diffeomorphisms

Intuition 3.7. Smooth maps between manifolds are the generalization of differentiable functions between Euclidean spaces. Diffeomorphisms are smooth maps that have smooth inverses, providing an equivalence between manifolds.

Definition 3.8. A **smooth map** between two manifolds M and N is a continuous function $f : M \to N$ such that for any charts (V, ψ) on N, and any chart (U, φ) on M, the composition $\psi \circ f \circ \varphi^{-1}$ is a smooth function between the open subsets of Euclidean spaces $\phi(U \cap f^{-1}(V))\mathbb{R}^m$ and $\psi(V) \subset \mathbb{R}^n$.

Definition 3.9. A **diffeomorphism** is a smooth map $f : M \to N$ between manifolds that has a smooth inverse. That is, f is a bijection, $f^{-1} : N \to M$ exists, and both f and f^{-1} are smooth maps. Two manifolds M and N are called **diffeomorphic** if there exists a diffeomorphism between them, meaning they are equivalent from the perspective of differential geometry.

Note 3.10. A diffeomorphism allows us to transfer the structure of one manifold to another smoothly. If M and N are diffeomorphic, they are "the same" as far as smooth structure is concerned, even if they may have different global geometries.

Examples:

- **Smooth Functions:** A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is a basic example of a smooth map. For instance, the function $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ is smooth as it is infinitely differentiable.

  A function $f : M^n \to \mathbb{R}$ is smooth if $f \circ \phi_\alpha^{-1}$ is smooth once restricted to $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$. We denote by $C^\infty(M)$ the set of smooth functions on M.

- **Smooth Curves:** A smooth curve in a manifold M is a smooth map $\gamma : I \to M$, where $I \subseteq \mathbb{R}$ is an open interval. For example, a smooth curve on the 2-dimensional sphere $\mathbb{S}^2$ could be given by $\gamma(t) = (\cos t, \sin t, 0)$, a smooth parametrization of a circle.

- **Rotation on the Torus:** Consider the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $f(\theta_1, \theta_2) = (\theta_1 + a, \theta_2 + b)$ for constants $a, b \in \mathbb{R}$. This map is a diffeomorphism as it is smooth and has a smooth inverse $f^{-1}(\theta_1, \theta_2) = (\theta_1 - a, \theta_2 - b)$.

3.3 Partition of unity

The concept of partition of unity is essential in differential geometry: it allows the construction of global objects on a manifold from local data defined on charts of an atlas. Given an atlas, a partition of unity always exists.

Definition 3.11. A **partition of unity** on a manifold M is a collection of smooth functions $\{\chi_i\}_{i \in I}$ on M such that:

- $0 \leq \chi_i(p) \leq 1$ for all $p \in M$ and for each $i \in I$.
- The support of each χ_i is contained within a chart of M, and for each point $p \in M$, there is a neighborhood where only finitely many χ_i are non-zero.
- The functions $\{\chi_i\}$ satisfy the property that, for all $p \in M$,

$$\sum_{i \in I} \chi_i(p) = 1.$$

Remark 3.12. The local finiteness is not always required and depends on the purpose.

4 Tangent bundle

The tangent bundle and vector bundles more generally are where (multi)-linear algebra takes place on a manifold.

4.1 Tangent bundle of a submanifold

Intuition 4.1. The tangent bundle of a submanifold consists of the tangent spaces of all points on the submanifold, considered as a subspace of the tangent bundle of the ambient manifold.

Definition 4.2. Let M be an n-dimensional manifold and $N \subset M$ be a k-dimensional submanifold. The **tangent bundle** TN of N is the collection of all tangent spaces T_pN at each point $p \in N$. Formally,

$$TN = \bigsqcup_{p \in N} T_pN,$$

where each T_pN is a k-dimensional subspace of the tangent space T_pM at p.

Note 4.3. The tangent bundle of N is a sub-bundle of the tangent bundle of M, meaning $TN \subset TM$ as vector bundles.

Example 4.4. Consider $\mathbb{S}^2 \subset \mathbb{R}^3$. The tangent bundle $T\mathbb{S}^2$ consists of all tangent planes to $\mathbb{S}^2$ at each point. Each $T_p\mathbb{S}^2$ is a 2-dimensional subspace of the 3-dimensional space $T_p\mathbb{R}^3$.

4.1.1 Tangent space of a level set

Definition 4.5. Let $M \subset \mathbb{R}^n$ be a submanifold defined as the level set of a smooth function $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$, i.e.,

$$M = \{p \in \mathbb{R}^n : F(p) = 0\}.$$

The **tangent space** T_pM at a point $p \in M$ is the kernel of the differential of F at p, that is,

$$T_pM = \ker(dF_p) = \left\{v \in T_p\mathbb{R}^n : dF_p(v) = 0\right\}.$$

Note 4.6. The differential dF_p at a point $p \in M$ is a linear map $dF_p : T_p\mathbb{R}^n \to \mathbb{R}^{n-k}$. The vectors in T_pM are precisely those in the null space of this map, meaning they are orthogonal to the gradients of the component functions of F.

Example 4.7. Consider the 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, which can be defined as the level set of the function $F(x, y, z) = x^2+y^2+z^2-1$. The tangent space $T_p\mathbb{S}^2$ at a point $p = (x_0, y_0, z_0)$ consists of all vectors $v \in \mathbb{R}^3$ such that $\nabla F(p)\cdot v = 0$, meaning v is orthogonal to the radial direction at p.

4.1.2 Tangent space of a parametrized submanifold

Intuition 4.8. When a submanifold is described using a parametrization, the tangent space at each point is spanned by the derivatives of the parametrization with respect to the local coordinates.

Definition 4.9. Let $N \subset \mathbb{R}^n$ be a k-dimensional submanifold parametrized by a smooth map $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$, where U is an open set. The **tangent space** $T_{\varphi(u)}N$ at a point $\varphi(u) \in N$ is spanned by the partial derivatives of the parametrization map φ, i.e.,

$$T_{\varphi(u)}N = \operatorname{span}\left\{\frac{\partial\varphi}{\partial u_1}(u), \dots, \frac{\partial\varphi}{\partial u_k}(u)\right\}.$$

Note 4.10. The tangent space at a point on a parametrized submanifold is a linear combination of the vectors obtained by differentiating the parametrization with respect to each of the local coordinates $u_1, \dots, u_k$. These vectors form a basis for the tangent space.

Example 4.11. Consider the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$, parametrized by $\varphi(\theta) = (\cos\theta, \sin\theta)$. The tangent space $T_{\varphi(\theta)}\mathbb{S}^1$ at a point $\varphi(\theta)$ is spanned by the derivative of φ with respect to θ, i.e., $T_{\varphi(\theta)}\mathbb{S}^1 = \operatorname{span}\{-\sin\theta, \cos\theta\}$.

4.2 Tangent space of a manifold

Intuition 4.12. The tangent space at a point on a manifold is the set of possible directions in which one can move from that point within the manifold.

Definition 4.13. Let M be a smooth manifold and $p \in M$. The **tangent space** T_pM at p is the vector space consisting of all equivalence classes of smooth curves through p. Formally, for a curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$, the equivalence class of γ defines a tangent vector $[\gamma] \in T_pM$. We say that $[\gamma_1] = [\gamma_2]$ if $\gamma_1'(0) = \gamma_2'(0)$.

Note 4.14. The dimension of the tangent space T_pM is the same as the dimension of the manifold M, i.e., $\dim(T_pM) = \dim(M)$. The tangent bundle TM is a manifold of dimension $2\dim M$.

Example 4.15.

- For $M = \mathbb{R}^n$, the tangent space at any point $p \in \mathbb{R}^n$ is naturally isomorphic to $\mathbb{R}^n$ itself.
- For the 2-sphere $\mathbb{S}^2$, the tangent space $T_p\mathbb{S}^2$ at a point p is a 2-dimensional plane tangent to the surface of the sphere at p.

4.3 Differential of a map

Intuition 4.16. The differential of a smooth map between manifolds determines how the map pushes forward tangent vectors from one manifold to another.

Definition 4.17. Let $f : M \to N$ be a smooth map between manifolds. The **differential** of f at a point $p \in M$, denoted df_p, is a linear map between tangent spaces:

$$df_p : T_pM \to T_{f(p)}N,$$

which pushes forward a tangent vector $v \in T_pM$ to a tangent vector $df_p(v) \in T_{f(p)}N$.

Note 4.18. The differential df_p is defined using the action of f on smooth curves. If $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve with $\gamma(0) = p$, then

$$df_p([\gamma]) = [f \circ \gamma].$$

Example 4.19.

- For $f : \mathbb{R}^2 \to \mathbb{R}^2$, the differential df_p is the Jacobian matrix of partial derivatives at p.
- For the projection map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, defined by $\pi(x, y, z) = (x, y)$, the differential $d\pi_p$ simply drops the third coordinate, projecting vectors onto the xy-plane.

4.4 Immersion, embedding and submanifolds

Definition 4.20. A smooth map $f : M \to N$ is called an **immersion** if its differential df_p is injective at every point $p \in M$. In other words, the map locally pushes forward distinct tangent vectors from M to distinct vectors in N.

Definition 4.21. A **submanifold** $N \subset M$ is a subset of a manifold M that is itself a manifold, and the inclusion map $i : N \to M$ is both an immersion and homeomorphism.

Note 4.22. An immersion does not need to be a homeomorphism onto its image, but an **embedding** is a stronger notion where the map is both an immersion and a homeomorphism onto its image.

Example 4.23.

- The inclusion map $i : \mathbb{S}^1 \to \mathbb{R}^2$, where $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$, is an immersion and also an embedding.
- The map $t \mapsto (t^2, t^3 - t)$ is an immersion but not an embedding.
- The map $t \mapsto (t^2, t^3)$ is not an immersion but it is a homeomorphism onto its image.

5 Sections of Vector Bundles and Vector Fields

A vector bundle is a collection of vector spaces parameterized smoothly by points on a manifold. A section of a vector bundle is a smooth choice of a vector from each fiber. Vector fields and tensors are a special case of sections, where the vector bundle is the tangent bundle of a manifold.

5.1 Vector Bundles

Intuition 5.1. A vector bundle assigns a vector space to each point on a manifold in a smooth manner. The idea is to generalize the notion of a product space like $M \times \mathbb{R}^n$, but allowing the vector space to vary smoothly as we move across the manifold.

Definition 5.2. Let M be a smooth manifold. A **vector bundle** $E \to M$ is a smooth manifold E together with a smooth surjection $\pi : E \to M$, such that for every point $p \in M$, the preimage $\pi^{-1}(p) = E_p$ is a vector space.

Additionally, close to any point $p \in M$, there exists an open neighborhood $U \subset M$ and a diffeomorphism $\phi = (\phi_1, \phi_2) : \pi^{-1}(U) \to U \times E_p$ so that,

- $\phi_1 = \pi_{|\pi^{-1}(U)}$, and
- for any $p' \in U$, $(\phi_2)_{|E_{p'}} : E_{p'} \to E_p$ is a linear isomorphism.

The local diffeomorphism ϕ is called a local **trivialization**.

Each E_p is called the **fiber** over p and the map π is called the **projection**. In specific contexts, M is called the **base**.

Local trivialization of the tangent bundle. Let M be a smooth n-dimensional manifold. Consider an open subset $U \subset M$ equipped with a smooth chart (coordinate map) $\varphi : U \to V \subset \mathbb{R}^n$, where $V = \varphi(U)$ is an open subset of $\mathbb{R}^n$. The chart φ allows us to construct a local trivialization of the tangent bundle TM over U as the pullback of the tangent bundle of $\mathbb{R}^n$ via φ.

Define the *pullback bundle* $\varphi^* T\mathbb{R}^n$ over U as:

$$\varphi^* T\mathbb{R}^n = \left\{ (p, w) \in U \times T_{\varphi(p)}\mathbb{R}^n \mid p \in U,\ w \in T_{\varphi(p)}\mathbb{R}^n \right\}.$$

This pullback bundle is a vector bundle over U with projection map $\pi' : \varphi^* T\mathbb{R}^n \to U$ given by $\pi'(p, w) = p$. Since at every point $\varphi(p)$, there is an identification $T_{\varphi(p)}\mathbb{R}^n \approx \mathbb{R}^n$, there is a natural identification $\varphi^* T\mathbb{R}^n \approx U \times \mathbb{R}^n$.

There is a natural vector bundle isomorphism between $TM|_U$ and $\varphi^* T\mathbb{R}^n$ given by the differential of φ:

$$\Phi : TM|_U \to \varphi^* T\mathbb{R}^n, \quad \Phi(p, v) = \left(p, d_p\varphi(v)\right).$$

This map Φ is linear on the fibers and smooth, making it a vector bundle isomorphism over U. Using $\varphi^* T\mathbb{R}^n \approx U \times \mathbb{R}^n$, this yields a local trivialization of TM over U.

This corresponds to the identification we used to prove that T_pM is a vector space, namely: $[\gamma]_p \mapsto [\varphi \circ \gamma]_{\phi(p)}$.

Example 5.3.

- The trivial vector bundle $M \times \mathbb{R}^n \to M$, where the fiber over each point $p \in M$ is $\mathbb{R}^n$.
- The **cotangent** bundle denoted T^*M is a vector bundle whose fiber at p is the space of linear forms on T_pM denoted $(T_pM)^*$.

- The most important vector bundles are those whose fibers are tensorial products of T_pM and $(T_pM)^*$ (say respectively m and n times). We will denote this bundle $TM^{\otimes m} \otimes T^*M^{\otimes n}$. On a tangent space T_pM, it represents n-linear maps from $(TpM)^n$ to $(TpM)^m$.

  Note: tensorial products represent multilinear operations, $\dim(V \otimes W) = \dim V \times \dim W$. This is different from $V \times W = V \oplus W$ where .

- More generally, any construction of a vector spaces from linear algebra yields a construction of new vector bundles by working fiber by fiber.

Remark 5.4. There exist other types of *fiber* bundles, e.g. circle bundles, sphere bundles, principal bundles (with Lie group structures on the fibers), where the fibers have something else than a linear structure. In any such situation, the maps $(\phi_2)_{|E_{p'}}$ preserve said structure. Many theories in physics, topology and geometry correspond to equations on fiber bundles, e.g. Hodge theory, Yang-Mills theory, harmonic spinors, Seiberg-Witten theory.

5.2 Sections of a vector bundle

Intuition 5.5. A section of a vector bundle is a smooth assignment of a vector in each fiber of the bundle over the manifold.

Definition 5.6. Let $E \to M$ be a vector bundle over a manifold M. A **section** of E is a smooth map $s : M \to E$ such that $\pi(s(p)) = p$ for every $p \in M$. In other words, for each p, $s(p) \in E_p$, where E_p is the fiber over p.

Note 5.7. The space of all smooth sections of E is often denoted by $\Gamma(E)$.

Example 5.8. A **smooth map** $f : M \to \mathbb{R}$, yields a section of the trivial line bundle $M \times \mathbb{R} \to M$ through $s(p) = (p, f(p))$.

5.3 Vector Fields, differential forms and tensors

Intuition 5.9. A vector field is a smooth assignment of a tangent vector at every point on a manifold. It is a section of the tangent bundle, giving a direction of flow on the manifold.

Definition 5.10. A **vector field** on a manifold M is a smooth map $X : M \to TM$ such that $X(p) \in T_pM$ for each $p \in M$. Equivalently, a vector field is a section of the tangent bundle TM.

Definition 5.11. A **differential 1-form** is a section of T^*M.

Definition 5.12. A **(m,n)-tensor** is a section of the vector bundle $TM^{\otimes m} \otimes T^*M^{\otimes n}$.

5.4 First order ODEs on manifolds

5.4.1 Flow of a vector field

Intuition 5.13. A first-order ordinary differential equation (ODE) on a manifold can be understood in terms of a vector field: the solution to such an ODE (the flow of the vector field) represents a curve on the manifold whose velocity at any point is given by the vector field at that point.

Definition 5.14. Let X be a smooth vector field on a manifold M. A **flow** of X is a smooth map $\varphi : I \times M \to M$, denoted by $\varphi_t(p)$, such that for each $p \in M$, the curve $\varphi_t(p)$ satisfies the first-order ODE

$$\frac{d}{dt}\varphi_t(p) = X(\varphi_t(p)).$$

Note 5.15. The flow $\varphi_t(p)$ is a curve on M starting at p, and at every point along the curve, its velocity is given by the vector field X. For small t, the flow moves in the direction of X, generating a trajectory.

Example 5.16. Consider the vector field $X(x, y) = (y, -x)$ on $\mathbb{R}^2$ which you will often find denoted as $X = y\partial_x - x\partial_y$. The flow of this vector field describes circular motion around the origin. The solution to the ODE

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -x(t))$$

is given by $(x(t), y(t)) = (x(0)\cos t - y(0)\sin t, x(0)\sin t + y(0)\cos t)$, describing rotation in $\mathbb{R}^2$.

5.4.2 Existence and Uniqueness of Solutions

Intuition 5.17. The classical Cauchy-Lipschitz theory of ODEs in $\mathbb{R}^n$ extends to manifolds.

Theorem 5.18 (Existence and Uniqueness)**.** Let X be a smooth vector field on a manifold M. For each point $p \in M$, there exists a unique maximal integral curve $\gamma : I \to M$ such that $\gamma(0) = p$ and

$$\frac{d}{dt}\gamma(t) = X(\gamma(t)).$$

Note 5.19. The maximal interval I depends on the behavior of the vector field. If X is complete, meaning its flow is defined for all time, then $I = \mathbb{R}$. Otherwise, I may be a finite interval.

Example 5.20. Consider the vector field $X(x, y) = (y, -x)$ on $\mathbb{R}^2$ again. The existence and uniqueness theorem ensures that for any initial point $(x_0, y_0) \in \mathbb{R}^2$, there is a unique solution to the ODE given by circular motion around the origin.

5.4.3 Diffeomorphisms from flows of vector fields

Intuition 5.21. A flow generated by a vector field can be used to construct a one-parameter family of diffeomorphisms on a manifold. These diffeomorphisms represent the "flow" of the manifold along the direction of the vector field over a fixed period of time.

Definition 5.22. Let X be a smooth vector field on a manifold M, and let φ_t denote the flow of X. Then, for each fixed $t \in \mathbb{R}$, the map $\varphi_t : M \to M$ defined by $\varphi_t(p) = \varphi(t, p)$ is called the **diffeomorphism generated by the flow** of the vector field X at time t.

Note 5.23. The diffeomorphism property holds as long as the flow exists for all time t. These diffeomorphisms form a one-parameter group, satisfying the following properties:

$$\varphi_0 = \mathrm{id}_M, \quad \varphi_{t+s} = \varphi_t \circ \varphi_s \quad \text{for all } t, s \in \mathbb{R}.$$

Example 5.24. Consider the vector field $X(x, y) = (y, -x)$ on $\mathbb{R}^2$, which generates the circular flow described earlier. For each fixed time t, the flow φ_t defines a rotation diffeomorphism of the plane, given by

$$\varphi_t(x, y) = (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t)),$$

which corresponds to a rotation by angle t in $\mathbb{R}^2$. These diffeomorphisms form a one-parameter group of rotations.

5.5 Vector fields and operations

5.5.1 Action of vector fields on functions and other vector fields

Intuition 5.25. A vector field X on a smooth manifold M can act on functions and other vector fields. The action of X on a smooth function f gives the directional derivative of f along X.

Definition 5.26 (Action of a vector field on a function)**.** Let X be a smooth vector field on a manifold M, and let $f : M \to \mathbb{R}$ be a smooth function. The action of X on f, denoted $X(f)$, is the function that assigns to each point $p \in M$ the derivative of f in the direction of X, i.e.,

$$X(f)(p) = \frac{d}{dt}\bigg|_{t=0} f(\varphi_t^X(p)),$$

where φ_t^X is the flow of X. This expression is also known as the **directional derivative** of f along X and measures how f changes as we move in the direction determined by X at each point.

5.5.2 Lie bracket of vector fields

Intuition 5.27. Vector fields can be added and scaled pointwise. More interestingly, we can compute the derivative of a vector field in the direction of another vector field, which is known as the Lie derivative.

Definition 5.28. Let X, Y be two vector fields on a manifold M. The **Lie bracket** or **Lie derivative** of X with respect to Y, denoted $[X, Y]$, is a new vector field defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)),$$

where f is a smooth function on M.

Note 5.29. The Lie bracket measures the failure of two vector fields to commute. In other words, it tells us how the flows generated by the vector fields interact.

Note 5.30. In local coordinates, if X and Y are expressed as

$$X = \sum_i X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_j Y^j \frac{\partial}{\partial x^j},$$

then the action of X on f is

$$X(f) = \sum_i X^i \frac{\partial f}{\partial x^i},$$

while the action of X on Y is given by

$$X(Y) = \sum_j \left(X^i \frac{\partial Y^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Example 5.31. Let $X(x, y) = \left(\frac{\partial}{\partial x}\right)$ and $Y(x, y) = \left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$ be vector fields on $\mathbb{R}^2$. Then

$$X(f)(x, y) = \frac{\partial f}{\partial x}(x, y),$$

and

$$X(Y)(x, y) = \frac{\partial}{\partial x}.$$

Example 5.32. For vector fields on $\mathbb{R}^2$, $X = \partial_x$ and $Y = \partial_y$, we have $[X, Y] = 0$, as the partial derivatives commute.

5.6 Tensors, tensors in coordinates, and Einstein's summation convention

Intuition 5.33. A tensor is a multilinear map that generalizes the concepts of scalars, vectors, and linear transformations. Tensors can be used to describe geometric and physical quantities that involve multiple directions, such as stress, curvature, and more. When expressed in coordinates, the tensor components follow certain transformation rules under changes of the coordinate system.

Definition 5.34. A **(m,n)-tensor** on a manifold M is a smooth section of the vector bundle $TM^{\otimes m} \otimes T^*M^{\otimes n}$, where TM is the tangent bundle and T^*M is the cotangent bundle. Such a tensor can be understood as taking n tangent vectors and m covectors as inputs and returns a scalar in a multilinear way.

Definition 5.35. A **tensor in coordinates** can be expressed in terms of a local basis. If $\{x^i\}$ are local coordinates on M, a tensor T of type (m, n) can be written as

$$T = T^{i_1 \cdots i_m}_{j_1 \cdots j_n} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_m}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_n},$$

where the components $T^{i_1 \cdots i_m}_{j_1 \cdots j_n}$ transform according to specific rules under a change of coordinates.

Definition 5.36. The **Einstein summation convention** is a notational shorthand used in tensor calculus. When an index appears as both an upper and a lower index in a term, it is implicitly summed over all values of that index. For example, in the expression

$$T^i_j v^j,$$

the index j is summed over all possible values, and the result is a vector field since it is of the form S^i meaning $S = S^i \frac{\partial}{\partial x^i}$.

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6 Metric Riemannian geometry

Riemannian geometry studies smooth manifolds equipped with a Riemannian metric, which allows for the measurement of lengths, angles, and volumes, and introduces concepts like curvature and geodesics.

6.1 Riemannian Manifolds

Intuition 6.1. A Riemannian manifold is a smooth manifold where each tangent space is equipped with an inner product that varies smoothly from point to point.

Definition 6.2. Let M be a smooth manifold. A **Riemannian metric** g on M is a smooth assignment to each point $p \in M$ of an inner product g_p on the tangent space T_pM. The pair (M, g) is called a **Riemannian manifold**.

Note 6.3. The smoothness condition means that for any smooth vector fields X, Y on M, the function

$$p \mapsto g_p(X_p, Y_p)$$

is smooth on M.

Theorem 6.4. Any differentiable manifold admits a Riemannian metric.

Example 6.5.

- The Euclidean space $\mathbb{R}^n$ with the standard dot product is a Riemannian manifold.
- The Euclidean space $\mathbb{R}^n$ with a smooth map $\mathbb{R}^n \mapsto Sym_2^+(n)$ into the space of symmetric positive definite matrices.

Note 6.6. General relativity studies **Lorentzian** manifolds, where the "metric" is not definite positive. It rather has one negative eigenvalue (in the direction of time) and 3 positive eigenvalues (in the direction of space). We will not discuss these geometries in this class, but most Riemannian objects have a Lorentzian counterpart with a closely related definition.

6.2 Induced Riemannian metric

Intuition 6.7. A submanifold of a Riemannian manifold inherits a Riemannian metric from the ambient space, allowing us to measure lengths and angles intrinsically.

Definition 6.8. Let (M, g) be a Riemannian manifold and $N \subset M$ be a submanifold. The **induced Riemannian metric** (sometimes called **first fundamental form**) h on N is defined by restricting g to T_pN:

$$h_p(u, v) = g_p(u, v), \quad \text{for all } u, v \in T_pN.$$

- The n-dimensional sphere $\mathbb{S}^n$ inherits a Riemannian metric from its embedding in $\mathbb{R}^{n+1}$.
- More generally, any submanifold M^n of $\mathbb{R}^N$, with $g_p = \langle \cdot, \cdot \rangle_{|T_pM}$, the restriction of the ambient standard dot-product $\langle \cdot, \cdot \rangle$ to the tangent spaces T_pM.

6.3 Orthonormal Bases and Frames

Given a Riemannian manifold (M, g), we say that a set of vectors $v_1, \ldots, v_n \in T_pM$ is **orthonormal** if:

$$g(v_i, v_j) = \delta_{ij}$$

An orthonormal frame on an open set $U \subset M$ is a collection of vector fields $E_1, \ldots, E_n$ such that:

$$g(E_i, E_j) = \delta_{ij} \quad \text{for all } p \in U$$

6.4 Metric geometry of Riemannian manifolds

6.4.1 Length of Curves and Distance

Intuition 6.9. The Riemannian metric allows us to measure lengths of tangent vectors and curves, defining a notion of distance on the manifold.

Definition 6.10. Let $\gamma : [a, b] \to M$ be a smooth curve on a Riemannian manifold (M, g). The **length** of γ is defined by

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt,$$

where $\|\gamma'(t)\| = \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}$.

Definition 6.11. The **distance** $d(p, q)$ between two points $p, q \in M$ is defined as the infimum of the lengths of all smooth curves connecting p and q:

$$d(p, q) = \inf_{\gamma} L(\gamma),$$

where γ ranges over all smooth curves with $\gamma(a) = p$ and $\gamma(b) = q$.

Remark 6.12. As a consequence, any Riemannian manifold is a **metric space**.

Definition 6.13. A curve that (locally) minimizes distances is a **geodesic**.

Example 6.14.

- In $\mathbb{R}^n$ with the standard metric, the distance between two points is the usual Euclidean distance.
- On the sphere $\mathbb{S}^2$, the distance between two points is the length of the shortest path along the sphere's surface, i.e., the great-circle distance.

6.4.2 Angles Between Geodesics and Vector Fields

Intuition 6.15. The Riemannian metric allows us to define angles between tangent vectors at a point, and by extension, angles between geodesics or vector fields. This generalizes the notion of angles from Euclidean geometry to curved spaces.

Definition 6.16. Let (M, g) be a Riemannian manifold, and let $X, Y \in T_pM$ be two tangent vectors at a point $p \in M$. The **angle** θ between X and Y is defined by

$$\cos(\theta) = \frac{g_p(X, Y)}{\|X\|\|Y\|},$$

where $\|X\| = \sqrt{g_p(X, X)}$ and $\|Y\| = \sqrt{g_p(Y, Y)}$.

Note 6.17. If γ_1 and γ_2 are two geodesics emanating from the same point $p \in M$, we define the angle between γ_1 and γ_2 to be the angle between their velocity vectors at p, i.e., $\gamma_1'(0)$ and $\gamma_2'(0)$.

Example 6.18. On the sphere $\mathbb{S}^2$, the angle between two great-circle geodesics is the same as the angle between their tangent vectors at the point where they intersect.

6.5 Curvature Bounds and the Toponogov Theorem in Synthetic Geometry

Intuition 6.19. In synthetic geometry, we study curvature by comparing the geometry of geodesic triangles in a manifold to those in spaces of constant curvature, without directly referencing the Riemann curvature tensor. Riemannian manifolds are purely seen as metric (length) spaces without any differential structure.

6.5.1 Triangles on Riemannian manifolds

Definition 6.20. Let (M^n, d) be a complete metric space where distances are measured using the Riemannian distance function induced by a smooth Riemannian metric. For any three points $p, q, r \in M$, a **geodesic triangle** consists of three minimizing geodesics $\gamma_{pq}, \gamma_{qr}, \gamma_{rp}$ connecting the points.

The idea of Alexandrov synthetic geometry is that the curvature of M can be studied by comparing the geometry of these geodesic triangles to model spaces of constant curvature K. The model spaces are:

- $\mathbb{R}^2$ for zero curvature (flat space),
- The sphere $\mathbb{S}^2_K$ for constant positive curvature $K > 0$,
- The hyperbolic plane $\mathbb{H}^2_K$ for constant negative curvature $K < 0$.

Example 6.21. In a manifold with constant positive curvature, such as the sphere $\mathbb{S}^2$, geodesic triangles have angle sums greater than π, while in a space of constant negative curvature, such as the hyperbolic plane $\mathbb{H}^2$, geodesic triangles have angle sums less than π.

6.5.2 Lower bounds on (sectional) curvatures

Theorem 6.22 (Toponogov's Theorem - Lower Bound)**.** Let (M, d) be a complete Riemannian manifold with sectional curvature bounded from below by K_0. Let $p, q, r \in M$ form a geodesic triangle, and let the corresponding comparison triangle in the model space $M^2_{K_0}$ have the same side lengths as $\triangle pqr$. Then the angles in the geodesic triangle $\triangle pqr$ are greater than or equal to the angles of the comparison triangle in $M^2_{K_0}$.

Remark 6.23. A theory of synthetic lower bounds on Ricci curvature has also been developed in the past two decades starting from ideas of Lott-Villani and Sturm. It is still a quite active field.

6.5.3 Upper bounds on (sectional) curvatures

Theorem 6.24 (Rauch comparison theorem - Upper Bound)**.** Let (M, d) be a complete Riemannian manifold with sectional curvature bounded from above by K_0. Let $p, q, r \in M$ form a geodesic triangle, and let the corresponding comparison triangle in the model space $M^2_{K_0}$ have the same side lengths as $\triangle pqr$. Then the angles in the geodesic triangle $\triangle pqr$ are less than or equal to the angles of the comparison triangle in $M^2_{K_0}$.

Intuition 6.25. An upper bound on sectional curvature ensures that geodesics diverge more than they would in flat space. If $K \leq K_0$, geodesic triangles in M are "thinner" than in flat space, meaning that their angles are smaller than those of a Euclidean or spherical triangle with the same side lengths.

6.6 Riemannian Volume Forms

Intuition 6.26. The Riemannian metric on a manifold not only allows for measuring lengths and angles but also provides a natural way to measure volumes.

Definition 6.27. Let (M, g) be an n-dimensional Riemannian manifold. The **Riemannian volume form** is a smooth n-form vol_g on M defined locally by

$$\mathrm{vol}_g = dv_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

where g_{ij} are the components of the metric in local coordinates, and $dx^1, dx^2, \ldots, dx^n$ are the standard coordinate differential forms.

Remark 6.28. The volume form vol_g provides a natural way to **integrate functions** over the manifold. For any smooth function $f \in C^\infty(M)$, its integral over M is given by

$$\int_M f\,\mathrm{vol}_g = \int_M f\sqrt{\det(g_{ij})}\,dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Strictly speaking, one integrates that way over every chart, i.e. for f supported in the chart. In order to integrate a function supported on the whole manifold (M, g), one decomposes

$$f = \sum_i \chi_i f,$$

according to a partition of the unity χ_i associated to a covering of M by charts. The sum of the local integrals of the $\chi_i f$ is the integral of f.

Example 6.29. On $\mathbb{R}^n$ with the standard Euclidean metric, the volume form is simply $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$, corresponding to the usual notion of volume in Euclidean space.

Note 6.30. The Riemannian volume form also allows for defining geometric invariants, such as the total volume of the manifold and the integral of specific curvature-related quantities. The L^2-norm of a vector field X over a Riemannian manifold (M, g) for instance is:

$$\|X\|^2_{L^2(g)} := \int_M g(X, X)\mathrm{vol}_g.$$

6.7 Riemannian Isometries

Intuition 6.31. An isometry is a *bijective* map between Riemannian manifolds that preserves the Riemannian metric.

Definition 6.32. Let (M, g) and (N, h) be two Riemannian manifolds. A smooth map $\phi : M \to N$ is called a **Riemannian isometry** if for all points $p \in M$ and tangent vectors $X, Y \in T_pM$,

$$h_{\phi(p)}(d\phi_p(X), d\phi_p(Y)) = g_p(X, Y).$$

If such a map exists, the manifolds M and N are said to be **isometric**.

Remark 6.33. A Riemannian isometry preserves all geometric structures that arise from the metric, such as lengths of curves, angles between vectors, and volumes. In particular, geodesics are mapped to geodesics under an isometry.

Example 6.34. The Euclidean space $\mathbb{R}^n$ with its standard metric has the group of isometries given by the Euclidean group, consisting of translations, rotations, and reflections.

Note 6.35. The group of all isometries of a Riemannian manifold forms a Lie group, called the **isometry group** of the manifold.

Remark 6.36. More generally, for a general $(0, m)$-tensor T, we define the **pull-back** by ϕ as

$$(\phi^* T)_p(V_1, \ldots, V_m) = T_{\phi(p)}(d\phi(V_1), \ldots, d\phi(V_m)).$$

6.8 Killing vector Fields

A vector field X on M is called a **Killing field** if its flow generates a 1-parameter family of isometries, i.e., the Lie derivative of the metric with respect to X vanishes:

$$\mathcal{L}_X g = 0$$

Remark 6.37. This is an infinitesimal condition, but as we have seen, the flow of X can always be defined, and integrating the identity $\mathcal{L}_X g = 0$ along the flow implies that the resulting family of diffeomorphisms $\phi_t^* g$ are isometries.

7 Derivatives on Riemannian manifolds

7.1 Affine Connections and Covariant Derivatives

Intuition 7.1. An affine connection provides a way to differentiate vector fields (and other elements of vector bundles) on a manifold, and provides a way to compare vectors at different points.

Definition 7.2. An **affine connection** ∇ on a manifold M is a map that assigns to each pair of vector fields X, Y a vector field $\nabla_X Y$, called the **covariant derivative** of Y along X, satisfying:

1. $\nabla_{fX+hY} Z = f\nabla_X Z + h\nabla_Y Z$,
2. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
3. $\nabla_X(fY) = f\nabla_X Y + (X(f))Y = f\nabla_X Y + df(X)Y$,

for all smooth functions f, h and vector fields X, Y, Z.

Remark 7.3. The tensor ∇Y is an element of $T^*M \otimes TM$.

7.2 Parallel Transport

Intuition 7.4. Parallel transport allows us to move vectors along a curve while keeping them parallel according to the manifold's connection. This is the geometric intuition behind a connection: it provides a correspondence between different fibers of a vector bundle–at least along curves.

A connection on a vector bundle provides a systematic way to compare vectors in the fibers over different points on the base manifold. In essence, it introduces a rule for "transporting" or "connecting" information between fibers of the vector bundle in a smooth, consistent manner.

Definition 7.5. Let $\gamma : [a, b] \to M$ be a smooth curve. A vector field $V(t)$ along γ is said to be **parallel** if:

$$\nabla_{\gamma'(t)} V(t) = 0.$$

Note 7.6. Parallel transport defines a linear isomorphism $P_{a\to b} : T_{\gamma(a)}M \to T_{\gamma(b)}M$, depending on the path γ.

Remark 7.7. The Lie derivative $\mathcal{L}_X Y$ measures the infinitesimal change of the vector field Y along the flow of the vector field X defined globally. The covariant derivative $\nabla_X Y$ measures the infinitesimal variation of the vector field Y in a "parallel transported sense" which is dependent on how nearby tangent spaces are related to each other through the connection.

7.3 The Levi-Civita Connection

Intuition 7.8. On a Riemannian manifold, there is a unique connection that preserves the metric and has no torsion; this is the Levi-Civita connection.

Theorem 7.9. Let (M, g) be a Riemannian manifold. There exists a unique affine connection ∇, called the **Levi-Civita connection**, satisfying:

1. **Metric compatibility**: $\nabla g = 0$, i.e.,

   $$\mathcal{L}_X[g(Y, Z)] = X[g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

   for all vector fields X, Y, Z.

2. **Torsion-free**: The torsion tensor $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ vanishes.

Proof. The Levi-Civita connection can be explicitly (hence uniquely) defined using the **Koszul formula**:

$$\begin{aligned}2g(\nabla_X Y, Z) = {} & X[g(Y,Z)] + Y[g(Z,X)] - Z[g(X,Y)] \\ & + g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X).\end{aligned}$$

This formula ensures both metric compatibility and vanishing torsion. □

Remark 7.10. The torsion-free condition provides a relationship between the connection and Lie brackets: $\nabla_X Y - \nabla_Y X = [X,Y]$.

7.3.1 Christoffel Symbols

In local coordinates, the connection ∇ can be written using the Christoffel symbols Γ^k_{ij} as:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

Using Koszul formula, the **Christoffel symbols** are given by:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

Note 7.11. Here, the notation g^{kl} means the coefficient (k,l) of the inverse of the matrix of the metric in coordinates $(g_{ij})_{i,j \in \{1,\dots,n\}}$. We will use this notation a lot, and it will be the tool to transform (a,b)-tensors to (c,d)-tensors if $a+b = c+d$.

Note 7.12. The Christoffel symbols are symmetric in their lower two indices:

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

7.3.2 Geodesics

Intuition 7.13. Geodesics are the "straightest" possible paths on a manifold, generalizing the notion of straight lines in Euclidean space to curved spaces. They can be defined using the Levi-Civita connection.

Definition 7.14. Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. A smooth curve $\gamma : [a,b] \to M$ is called a **geodesic** if its acceleration vector is zero with respect to ∇:

$$\nabla_{\gamma'} \gamma' = 0,$$

where γ' is the velocity vector of γ.

Note 7.15. This equation will be justified later: geodesics are critical points of the length functional.

Note 7.16. Geodesics satisfy the **geodesic equation** in local coordinates (x^i):

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0,$$

where Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection.

Example 7.17.

- In $\mathbb{R}^n$ with the standard metric, geodesics are straight lines, as the Christoffel symbols vanish.
- On the sphere $\mathbb{S}^n$, geodesics are great circles, which are the shortest paths between two points on the sphere.

Example 7.18. Geodesics on a Sphere:
Consider the 2-sphere $\mathbb{S}^2$ embedded in $\mathbb{R}^3$. In spherical coordinates (θ, ϕ), the metric is:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

The geodesic equations become:

$$\frac{d^2\theta}{dt^2} - \sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 = 0,$$
$$\frac{d^2\phi}{dt^2} + 2\cot\theta\frac{d\theta}{dt}\frac{d\phi}{dt} = 0.$$

These equations describe great circles on the sphere.

8 Levi-Civita connection on submanifolds and Second fundamental form

8.1 Induced Levi-Civita connection

Intuition 8.1. When a submanifold is embedded in a Riemannian manifold, it inherits a Riemannian metric from the ambient space. The Levi-Civita connection on the submanifold is related to that of the ambient manifold but must account for the curvature induced by the embedding.

Definition 8.2. Let $(\bar{M}, \bar{g})$ be a Riemannian manifold and $M \subset \bar{M}$ a submanifold with the induced metric $g = \bar{g}|_M$. The **Levi-Civita connection** ∇ on M is defined by projecting the ambient Levi-Civita connection $\bar{\nabla}$ onto M:

$$\nabla_X Y = (\bar{\nabla}_X Y)^\top,$$

for all vector fields X, Y tangent to M, where $(\cdot)^\top$ denotes the projection onto the tangent bundle TM.

Note 8.3. The projection $(\bar{\nabla}_X Y)^\top$ ensures that ∇ is compatible with the induced metric g and is torsion-free, thus satisfying the properties of the Levi-Civita connection on M.

8.2 Second Fundamental Form

Intuition 8.4. The second fundamental form quantifies how a submanifold bends within the ambient manifold. It measures the failure of the ambient connection to preserve tangency when differentiating tangent vector fields along the submanifold.

Definition 8.5. The **second fundamental form** II of M (the submanifold) in $\bar{M}$ (the ambient spaces) is the symmetric bilinear form defined by:

$$\mathrm{II}(X, Y) = (\bar{\nabla}_X Y)^\perp,$$

for all vector fields X, Y tangent to M, where $(\cdot)^\perp$ denotes the projection onto the normal bundle $T^\perp M$.

Note 8.6. The second fundamental form II takes values in the normal bundle $T^\perp M$ and encodes the extrinsic curvature of M within $\bar{M}$.

Proposition 8.7. The second fundamental form is symmetric:

$$\mathrm{II}(X, Y) = \mathrm{II}(Y, X),$$

for all tangent vector fields X, Y on M.

Proof. Since $\bar\nabla$ is torsion-free and $\bar g$ is compatible with $\bar\nabla$, we have:

$$\mathrm{II}(X,Y) = (\bar\nabla_X Y)^\perp = (\bar\nabla_Y X)^\perp = \mathrm{II}(Y,X).$$

$\square$

Note 8.8. In the case of hypersurfaces, once a unit normal vector is chosen (depends on conventions!), at least locally, the second fundamental form can be identified with a single real number:

$$\mathrm{II}(X,Y) =: A(X,Y)\vec{n}.$$

The eigenvalues of A restricted to a given tangent plane T_pM are called the **principal curvatures**. They are the curvatures of specific curves drawn on the manifold M.

8.3 Mean Curvature and Shape Operator

We restrict our attention to *hypersurfaces* here. There are vector versions of the mean curvature in higher codimensions, which we will not discuss.

Definition 8.9. The **mean curvature vector** H of M in $\bar M$ is defined as the trace of the second fundamental form:

$$H = \sum_{i=1}^n A(e_i,e_i) \in \mathbb{R},$$

where $\{e_1,\dots,e_n\}$ is an orthonormal basis of T_pM at point $p \in M$.

This is the sum of the principal curvatures.

Remark 8.10. These objects depend on a lot of conventions, the sign of H depends upon conventions of orientation, e.g. is the normal of a surface inward or outward, some people also define $H = \frac{1}{n}\sum_{i=1}^n A(e_i,e_i)$. Be very careful! Note that the vector $H\vec{n}$ is independent on the choice of $\vec{n}$.

Example 8.11. Depending upon conventions.

- The mean curvature of a plane is zero.
- The mean curvature of a cylinder $\mathbb{R}\times R\mathbb{S}^1$ of radius R is positive, equal to $\frac{1}{R}$.
- The mean curvature of a sphere $R\mathbb{S}^n \subset \mathbb{R}^{n+1}$ of radius R is positive, equal to $\frac{n}{R}$.

Intuition 8.12. The mean curvature of a hypersurface measures the variations of its volume along normal perturbations. **Minimal surfaces** have vanishing mean curvature.

8.4 Gaussian curvature

In the specific case of surfaces inside $\mathbb{R}^3$, a central quantity is that of the Gaussian curvature.

Definition 8.13. The **Gaussian curvature** of a submanifold $M^2 \subset \mathbb{R}^3$ is the determinant of A, or the product of the (two) principal curvatures.

Remark 8.14. It is independent on the convention for $\vec{n}$.

Example 8.15.

- The Gaussian curvature of a plane is zero.
- The Gaussian curvature of a cylinder is zero.
- The Gaussian curvature of a sphere of radius R is positive, equal to R^{-2}.
- The Gaussian curvature of a saddle is negative.

8.5 Geodesics on Submanifolds

Intuition 8.16. Geodesics on a submanifold M of an ambient Riemannian manifold $(\bar{M}, \bar{g})$ are curves that locally minimize distance on M. While geodesics in the ambient manifold $\bar{M}$ are characterized by vanishing acceleration with respect to the ambient Levi-Civita connection $\bar{\nabla}$, geodesics on M must respect the constraint that their velocity and acceleration remain tangent to M. The notion of vanishing projected acceleration offers an elegant way to express this constraint.

Definition 8.17. A smooth curve $\gamma : I \to M$ is a **geodesic** on the submanifold M if its acceleration with respect to the Levi-Civita connection ∇ on M vanishes, i.e.,

$$(\bar{\nabla}_{\dot{\gamma}}\dot{\gamma})^{\top} = \nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

This condition is interpreted as the vanishing of the projection of the ambient acceleration $\bar{\nabla}_{\dot{\gamma}}\dot{\gamma}$ onto the tangent bundle of M being zero.

8.5.1 Geodesics and the Second Fundamental Form

Intuition 8.18. When considering geodesics on M, the second fundamental form measures how the ambient geodesics deviate from M. From this perspective, the second fundamental form in the direction $\dot{\gamma}$, $\mathrm{II}(\dot{\gamma}, \dot{\gamma})$, can be interpreted as the minimal necessary acceleration for a curve with speed $\dot{\gamma}$ to stay on the manifold.

Proposition 8.19. Let $\gamma : I \to M$ be a geodesic on the submanifold M. Then the second fundamental form evaluated along γ satisfies:

$$\mathrm{II}(\dot{\gamma}, \dot{\gamma}) = (\bar{\nabla}_{\dot{\gamma}}\dot{\gamma})^{\perp}.$$

In particular, if γ is also a geodesic in $\bar{M}$, then $\mathrm{II}(\dot{\gamma}, \dot{\gamma}) = 0$, meaning γ experiences no normal acceleration.

Remark 8.20. The presence of non-zero normal acceleration, as captured by $\mathrm{II}(\dot{\gamma}, \dot{\gamma})$, reflects the extrinsic curvature of M in $\bar{M}$.

9 Gauss Equations for Hypersurfaces

Intuition 9.1. The Gauss and Codazzi equations provide fundamental relationships between the intrinsic geometry of a hypersurface and the geometry of the ambient space. These equations connect the Riemann curvature tensor of the hypersurface with that of the surrounding manifold, accounting for the second fundamental form.

Note 9.2. On (oriented) hypersurfaces, once a normal vector field $\vec{n}$ has been chosen, one abusively identifies the normal vector $II(X, Y) = A(X, Y)\vec{n}$ and the value $A(X, Y)$. Its sign depends upon the convention chosen for the normal vector $\vec{n}$.

9.1 Gauss Equation for Hypersurfaces

For a hypersurface M^n embedded in $\mathbb{R}^{n+1}$, the **Gauss equation** relates the intrinsic Riemann curvature tensor R of the hypersurface to the second fundamental form II and the curvature $\bar{R}$ of the ambient space $\mathbb{R}^{n+1}$. In the case of a flat ambient space, $\bar{R} = 0$, and the Gauss equation becomes:

$$g(R(X, Y)Z, W) = \bar{g}(\mathrm{II}(X, Z), \mathrm{II}(Y, W)) - \bar{g}(\mathrm{II}(Y, Z), \mathrm{II}(X, W)).$$

Here, g denotes the induced metric on M, and $\bar{g}$ is the metric of the ambient space.

Note 9.3. For now, as a first contact, we take this formula as a definition for the Riemann curvature of a hypersurface.

Note 9.4. Once again, there are different conventions, in some books, $g(R(X, Y)Z, W)$ might be $g(R(X, Y)W, Z)$ in others–changing the sign.

9.2 Ricci and Scalar Curvatures on Hypersurfaces

Intuition 9.5.

- The **Riemann curvature tensor** R on the hypersurface M measures how much a vector changes when parallel transported around a small loop, it controls the geometry at the level of curves. It essentially contains the information of the Gaussian curvature of all geodesic planes inside the manifold.
- The **Ricci curvature** $\mathrm{Ric}(X, Y)$ is an average of the Riemann curvature tensor in specific directions. It describes how volumes of small geodesic cones around geodesics (hence small balls) in M deviate from those in flat space. Upper and lower bounds on Ricci provide strong global control of the global geometry.
- The **Scalar curvature** S locally controls the volume of small balls in a manifold. It has some nontrivial and poorly understood global consequences (e.g. positive mass theorem in General Relativity). I physics, it is often called the **Ricci scalar**.

For a hypersurface M^n embedded in $\mathbb{R}^{n+1}$, we can compute the Ricci and scalar curvature in terms of the second fundamental form and the principal curvatures of the hypersurface.

Definition 9.6. The **Ricci curvature** $\mathrm{Ric}(X, Y)$ of the hypersurface can be computed as the trace of the Riemann curvature tensor with respect to an orthonormal basis $\{e_1, \dots, e_n\}$ tangent to M. It is given by:

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} g(R(X, e_i)e_i, Y).$$

Using the Gauss equation for the Riemann curvature, we can express the Ricci curvature in terms of the second fundamental form: if the ambient space is flat, then

$$\mathrm{Ric}(X, Y) = \mathrm{Tr}(\mathrm{II}) \cdot \mathrm{II}(X, Y) - \mathrm{II}^2(X, Y),$$

where $\mathrm{II}^2(X, Y)$ denotes the action of the second fundamental form squared.

Definition 9.7. The **scalar curvature** (or **Ricci scalar** in physics) S of the hypersurface is the trace of the Ricci tensor, which can be computed as:

$$S = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i),$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis. In terms of the second fundamental form II, the scalar curvature can be written as:

$$S = (\mathrm{Tr}(\mathrm{II}))^2 - \|\mathrm{II}\|^2,$$

where $\|\mathrm{II}\|^2$ denotes the squared norm of the second fundamental form.

Example 9.8 (Example: Sphere in $\mathbb{R}^{n+1}$)**.** Consider the standard n-sphere S^n embedded in $\mathbb{R}^{n+1}$. The second fundamental form of S^n is given by:

$$\mathrm{II}(X, Y) = g(X, Y),$$

where g is the round metric of S^n. The Gauss equation simplifies to:

$$g(R(X, Y)Z, W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W),$$

showing that in this specific situation, the Riemann curvature tensor can be expressed without referring to the first and second derivatives of the metric.

Ricci Curvature: For the n-dimensional sphere S^n, the Ricci curvature is constant and proportional to the metric. Specifically, for any tangent vectors X, Y on S^n, the Ricci curvature is given by:

$$\mathrm{Ric}(X, Y) = (n - 1)g(X, Y).$$

This shows that the Ricci curvature of the n-sphere is uniform and depends on the dimension of the sphere.

Scalar Curvature: The scalar curvature S of the n-sphere is the trace of the Ricci curvature. Since the Ricci curvature is proportional to the metric, the scalar curvature is constant and is given by:

$$S = n(n - 1).$$

Thus, the scalar curvature of the n-sphere depends only on its dimension and reflects the fact that the sphere has constant positive curvature.

Note 9.9. The *conventions* are typically uniform in the literature for Scalar curvature and Ricci curvature. The *notations* for all of the different curvature tensors vary widely.

10 Riemann curvature tensor and sectional curvatures

Intuition 10.1. The curvature tensor measures how much the manifold deviates from being flat by quantifying the failure of covariant derivatives to commute.

Definition 10.2. The **Riemann curvature tensor** R is a $(1,3)$-tensor defined for vector fields X, Y, Z by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Note 10.3. Some references sometimes consider the opposite formula.

Note 10.4. The term "Riemann tensor" is often abusively used for different types of tensors:

- as a $(1,3)$ tensor taking three tangent vectors as arguments to give one vector

  $$(X,Y,Z) \mapsto R(X,Y)Z,$$

  or $R^i{}_{jkl}$ in components: $(R(X,Y)Z)^i = R^i{}_{jkl} Z^j X^k Y^l$.
- as a $(0,4)$ tensor taking four tangent vectors as arguments to give one scalar

  $$(X,Y,Z,W) \mapsto g(R(X,Y)Z, W),$$

  or R_{ijkl} in components, and
- as a $(2,2)$ tensor taking two vectors as arguments to give two vectors

  $$(X,Y) \mapsto R(X,Y),$$

  or $R^{ij}{}_{kl}$ in components, and

Note 10.5. The Riemann tensor has specific symmetries:

- **Antisymmetry in the first two indices:**

  $$R(X,Y)Z = -R(Y,X)Z,$$

  or equivalently, in terms of the components:

  $$R_{ijkl} = -R_{jikl}.$$

- **Antisymmetry in the last two indices:**

  $$g(R(X,Y)Z, W) = -g(R(X,Y)W, Z),$$

  or, in components:

  $$R_{ijkl} = -R_{ijlk}.$$

- **Symmetry of the interchange of pairs:**

  $$g(R(X,Y)Z, W) = g(R(Z,W)X, Y),$$

  or in components:

  $$R_{ijkl} = R_{klij}.$$

- **First Bianchi identity:** This identity expresses the cyclic sum of the Riemann tensor over its first three indices as zero:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

  or in components:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0.$$

Definition 10.6. The **sectional curvature** K of a two-dimensional plane $\sigma \subset T_pM$ spanned by u, v is given by:

$$K(\sigma) = \frac{g(R(u,v)v,u)}{\|u\|^2\|v\|^2 - g(u,v)^2}.$$

Note 10.7. If $\|u\| = \|v\| = 1$ and $g(u,v) = 0$, then

$$K(\sigma) = g(R(u,v)v,u).$$

Note 10.8. This relationship between sectional curvatures and Riemannian tensor depends on conventions. Everyone agrees that the round unit sphere should have sectional curvatures equal to 1.

Example 10.9.

- **Euclidean space** $\mathbb{R}^n$: The curvature tensor R vanishes identically, so $K(\sigma) = 0$ for all σ.
- **Sphere** $\mathbb{S}^n$: The sectional curvature is constant and positive, $K(\sigma) = 1$ (assuming the sphere has radius 1).

10.1 Raising and lowering indices of tensors using the metric

Given an (a,b)-tensor, there is a natural way to obtain a (c,d)-tensor using the metric by raising and lowering specific indices.

Intuition 10.10. On Euclidean space, one can associate a linear form to a vector $\vec{v}$ from the formula

$$\vec{w} \mapsto \vec{v} \cdot \vec{w} \in \mathbb{R}.$$

Similarly, a linear form is uniquely associated to a vector so that the above formula holds.

10.2 Raising and Lowering Indices

Intuition 10.11. In Riemannian geometry, raising and lowering indices refers to converting between covariant and contravariant tensors using the metric. This operation allows for flexibility in manipulating tensorial expressions, especially when working with quantities like the Riemann curvature tensor.

If $p \in U \cap U'$, and if (U,ϕ) and (U',ϕ') are two charts, then we have two representations of p, namely: $x = \phi(p)$ and $x' = \phi'(p)$, i.e. $x' = \phi' \circ \phi^{-1}$. The coordinates of tensors change in specific ways when going from x to x'.

Definition 10.12. A **contravariant vector** (or simply vector) is an object that transforms under a change of coordinates as follows:

$$V'^i = \frac{\partial x'^i}{\partial x^j} V^j,$$

where x^i and x'^i represent two different coordinate systems. That is, under a change of coordinates from $x = (x^1, \dots, x^n)$ to $x' = (x'^1, \dots, x'^n)$, the components of a contravariant vector transform by the Jacobian matrix of the coordinate transformation.

A **covariant vector** (or covector, or 1-form) transforms in the opposite manner:

$$\omega'_i = \frac{\partial x^j}{\partial x'^i} \omega_j,$$

where the components of the covariant vector transform by the inverse of the Jacobian matrix.

Remark 10.13. The terms "contravariant" and "covariant" refer to the way these objects transform relative to a change of coordinates. A contravariant vector transforms by the Jacobian matrix of the coordinate transformation, in a direction opposite (contra) to the way the coordinates themselves change. In contrast, a covariant vector transforms by the inverse of the Jacobian, in the same (co) direction as the coordinate change. Contravariant vectors have raised indices, while covariant vectors have lowered indices.

Definition 10.14. Given a **metric** g on a manifold M, we can raise or lower the index of a tensor using the metric and its inverse. For example, for a vector field X, we can obtain a corresponding covector (1-form) $X^\flat$ by:

$$X^\flat = g(X, \cdot),$$

which satisfies $X^\flat(Y) = g(X, Y) \quad \text{for any vector field } Y.$

Similarly, for a covector ω, we can raise its index using the inverse of the metric g^{-1} to get a vector field $\omega^\sharp$ defined by:

$$g(\omega^\sharp, Y) = \omega(Y).$$

Note 10.15. This process of raising and lowering indices is a direct consequence of the non-degenerate pairing given by the metric g. The metric allows vectors and covectors to be paired and manipulated in this way.

For tensors with multiple indices, such as the curvature tensor, we can raise and lower any of the indices using the metric. For instance, starting with the Riemann curvature tensor R as a $(1,3)$-tensor, we can lower the first index to form a fully covariant $(0,4)$-tensor:

$$R_{ijkl} = g_{im} R^m{}_{jkl}.$$

Similarly, one has

$$R^i{}_{jkl} = g^{im} R_{mjkl}.$$

Definition 10.16. The **fully covariant curvature tensor** R_{ijkl} is obtained by lowering the upper index of R^i_{jkl} using the metric g. This fully covariant form often simplifies calculations involving symmetries and contractions.

Note 10.17. The symmetries of the curvature tensor are often more easily expressed when working with fully covariant or fully contravariant versions, depending on the problem at hand. This is why raising and lowering indices is a key tool in computations involving curvature.

Example 10.18.

- For a vector field X, lowering the index results in a covector $X^\flat$. If X is given by X^i in coordinates, then $X_i = g_{ij} X^j$.
- For a 1-form ω, raising the index gives a vector field $\omega^\sharp$. In coordinates, if ω_i is the covector, then the corresponding vector field is given by $\omega^i = g^{ij} \omega_j$.

Definition 10.19. The **Riemannian gradient** of a smooth function u is the vector field $\nabla u = (du)^\sharp$, the unique one satisfying $du(X) = g(\nabla u, X)$. The gradient depends on the metric g while du *does not*.

10.3 Exponential Map and Local Geometric Properties

Definition 10.20. The **exponential map** is a fundamental tool in Riemannian geometry, mapping tangent vectors at a point to points on the manifold. Given a point $p \in M$ and a tangent vector $v \in T_pM$, the exponential map $\exp_p$ is defined by:

$$\exp_p(v) = \gamma_v(1),$$

where γ_v is the unique geodesic starting at p with initial velocity v.

The exponential map provides a natural atlas, allowing us to describe neighborhoods of points in terms of tangent vectors. It depends on a Riemannian metric.

Definition 10.21. The inverse of the map $v \mapsto \exp_p(v)$ is a diffeomorphism on its image as long as v is small enough. This provides so-called **normal coordinates** on small geodesic balls around p.

For each $p \in M$ and small $r > 0$, the exponential map $\exp_p$ is a diffeomorphism on the ball $B_r(0) \subset T_pM$. This means that the exponential map sends balls in T_pM to geodesic balls in M (defined by the metric). Moreover, the topology induced by the distance function d on M agrees with the natural topology of M.

The question arises: how far can the exponential map define a chart? The answer depends on the presence of *conjugate points* and the *injectivity radius*.

Definition 10.22. A point q is conjugate to p along a geodesic γ if there exists a non-trivial Jacobi field along γ that vanishes at both p and q. The existence of conjugate points signals that the geodesic is no longer minimizing beyond q. The **conjugate radius** of a manifold depends on the sectional curvatures as we will see.

The **injectivity radius** at a point p is the largest radius such that the exponential map is a diffeomorphism on the ball of this radius in T_pM. If the injectivity radius is finite, geodesics may fail to be globally minimizing beyond this radius, leading to cut points.

Example 10.23.

- The conjugate and injectivity radii of the flat plane are infinite. More precisely, in $\mathbb{R}^n$, one has $\exp_x(X) = x + X$ since the geodesics are the straight lines. The injectivity radius is $+\infty$.
- The conjugate radius of a round cylinder $\mathbb{S}^1 \times \mathbb{R} \subset \mathbb{R}^3$ is infinite, but its injectivity radius is π.
- The conjugate radius of the round sphere is π.

11 Variations of curves and Jacobi fields

Let $\gamma : [a, b] \to M$ be a smooth curve. A **variation** of γ is a smooth map $F : (-\epsilon, \epsilon) \times [a, b] \to M$ such that $F(0, t) = \gamma(t)$. We say that the variation is **proper** if $F(s, a) = \gamma(a)$ and $F(s, b) = \gamma(b)$ for all s.

A variation of γ can be thought of as a one-parameter family of curves $\gamma_s(t) = F(s, t)$. The energy of a piecewise smooth curve $\gamma : [a, b] \to M$ is given by:

$$E(\gamma) = \frac{1}{2} \int_a^b g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\, dt.$$

11.1 First Variation of Energy

Let $\gamma_s(t) =: F(s, t)$ be a smooth variation of a curve $\gamma = F(0, t)$. The **variation vector field** $V : [a, b] \to TM$ is defined by:

$$V(t) = \frac{\partial}{\partial s}\Big|_{s=0} F(s, t) = d_{(0,t)}F(1, 0) = d_{(0,t)}F(\partial_s).$$

We also denote $\dot{\gamma}_s(t_0) = \frac{\partial}{\partial t}\Big|_{t=t_0} F(s, t) = d_{(s,t_0)}F(0, 1) = d_{(s,t_0)}F(\partial_t)$.

Remark 11.1. These two vectors commute since following s then t or the opposite yields the same end point $F(s, t)$, and consequently $\nabla_V \dot{\gamma}_s = \nabla_{\dot{\gamma}_s} V$ this can be see from the torsion-free condition and $[\partial_s, \partial_t] = 0$

$$\nabla_{\partial_s F}\partial_t F - \nabla_{\partial_t F}\partial_s F = [\partial_s F, \partial_t F] = dF\left(\left[\partial_s, \partial_t\right]\right) = 0. \quad \square$$

(as an exercise, you can also verify this directly in coordinates).

Since the connection is metric, we have

$$\begin{aligned}
\partial_{s|s=0}(g_{\gamma_s(t)}(\dot\gamma_s,\dot\gamma_s)) &= V(g(\dot\gamma_s,\dot\gamma_s))\\
&= 2(g(\nabla_V\dot\gamma,\dot\gamma))\\
&= 2(g(\nabla_{\dot\gamma}V,\dot\gamma))\\
&= -2(g(V,\nabla_{\dot\gamma}\dot\gamma)) + 2\dot\gamma(g(V,\dot\gamma))\\
&= -2(g(V,\nabla_{\dot\gamma}\dot\gamma)) + 2\partial_t g_{\gamma(t)}(V(t),\dot\gamma(t)).
\end{aligned}$$

The first variation of the energy is therefore given by:

$$\frac{d}{ds}\bigg|_{s=0} E(\gamma_s) = \frac{1}{2}\int_a^b \partial_{s|s=0}(g_{\gamma_s(t)}(\dot\gamma_s,\dot\gamma_s))dt = -\int_a^b (g(V,\nabla_{\dot\gamma}\dot\gamma))\,dt + \left[g_{\gamma(t)}(V(t),\dot\gamma(t))\right]_a^b.$$

Thus, a curve γ is a critical point of the energy functional with fixed end points (i.e. the above vanishes for all V with $V(a)=0$ and $V(b)=0$) if and only if $\nabla_{\dot\gamma}\dot\gamma = 0$, i.e. at geodesics.

Remark 11.2. Similarly, the critical points of the length functional are geodesics and their reparametrizations.

It might feel easier to see using the map F

$$\begin{aligned}
\frac{dE(\gamma_s)}{ds} &= \frac{1}{2}\int_a^b \frac{\partial}{\partial s}\langle\dot\gamma_s(t),\dot\gamma_s(t)\rangle\,dt = \frac{1}{2}\int_a^b \nabla_{\partial_s F}\langle\partial_t F,\partial_t F\rangle\,dt\\
&= \int_a^b \langle\nabla_{\partial_s F}\partial_t F,\partial_t F\rangle\,dt = \int_a^b \langle\nabla_{\partial_t F}\partial_s F,\partial_t F\rangle\,dt\\
&= \int_a^b \frac{\partial}{\partial t}\langle\partial_s F,\partial_t F\rangle\,dt - \int_a^b \langle\partial_s F,\nabla_{\partial_t F}\partial_t F\rangle\,dt\\
&= \langle\partial_s F,\partial_t F\rangle(b,s) - \langle\partial_s F,\partial_t F\rangle(a,s) - \int_a^b \langle\partial_s F,\nabla_{\partial_t F}\partial_t F\rangle\,dt.
\end{aligned}$$

11.2 Jacobi Fields

Now suppose γ is a geodesic and F is a geodesic variation of γ, i.e. each $\gamma_s = F(s,\cdot)$ is a geodesic. Let J be its variation field. Then

$$\nabla_{\partial_t F}\partial_t F = \nabla_{\dot\gamma_s}\dot\gamma_s = 0,$$

and since $[\partial_t F,\partial_s F] = dF\left[\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right] = 0$, we get

$$\nabla_{\partial_t F}\nabla_{\partial_t F}\partial_s F = \nabla_{\partial_t F}\nabla_{\partial_s F}\partial_t F = -\nabla_{\partial_s F}\nabla_{\partial_t F}\partial_t F + \nabla_{\partial_t F}\nabla_{\partial_s F}\partial_t F + \nabla_{[\partial_s F,\partial_t F]}\partial_t F = -R(\partial_s F,\partial_t F)\partial_t F.$$

Taking $s=0$, we see

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J + R(J,\dot\gamma)\dot\gamma = 0.$$

Definition 11.3. A vector field X along a geodesic γ is called a **Jacobi field** if

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}X + R(X,\dot\gamma)\dot\gamma = 0.$$

Example 11.4. Let γ be a geodesic.

1. The translation $X = \dot\gamma$ is a Jacobi field.

2. The reparametrization with constant speed $X = t\dot{\gamma}$ is a Jacobi field.

Theorem 11.5. Let $\gamma : [a, b] \to M$ be a geodesic. Then for any $X_{\gamma(a)}, Y_{\gamma(a)} \in T_{\gamma(a)}M$, there exists a unique Jacobi field X along γ such that

$$X(a) = X_{\gamma(a)} \quad \text{and} \quad \nabla_{\dot{\gamma}(a)} X = Y_{\gamma(a)}.$$

Proof. Assume that γ is parametrized by arc length. Let $(e_i(t))_i$ be an orthonormal basis at each point $\gamma(t)$, with each $e_i(t)$ obtained by parallel transport $\nabla_{\dot{\gamma}(t)} e_i(t) = 0$ along γ, and such that $e_1(t) = \dot{\gamma}(t)$. For a vector field $X = X^i(t)e_i(t)$ along γ,

$$\nabla_{\dot{\gamma}} X = \dot{X}^i(t)e_i(t) \quad \text{and} \quad \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X = \ddot{X}^i(t)e_i(t).$$

It follows that the Jacobi field equation becomes

$$\ddot{X}^i(t)e_i(t) + X^i(t)R^j{}_{11i}e_j(t) = 0,$$

which is equivalent to

$$\ddot{X}^i(t) + X^j(t)R^i{}_{11j} = 0, \quad 1 \le i \le m.$$

This is a system of second-order linear ODEs. The claim now follows from basic ODE theory. ☐

Corollary 11.6. The set of Jacobi fields along γ is a linear space of dimension $2m$.

Corollary 11.7. If $X(t)$ is a non zero Jacobi field along γ, then its zeroes are discrete. If $R^i_{11j} \leqslant 0$ along the curve, then a non zero Jacobi field has at most one zero.

Yet another corollary can be seen as a way to prove that the small **geodesic spheres** $S_r(p) = \{x \in M, d_g(p, x) = r\}$ for $r > 0$ small enough are orthogonal to the geodesics emanating from p.

Corollary 11.8. If for a Jacobi field X along a geodesic γ, one has $X(0) = 0$ and $\nabla_{\dot{\gamma}} X(0) \perp \dot{\gamma}(0)$. Then, $X(t) \perp \dot{\gamma}(t)$.

Proof. Because of the symmetries of the Riemann tensor, one has $R^j{}_{11}1 = 0$ for any i, hence $\ddot{X}^1(t) = 0$ for every t. Since by assumption $X^1(0) = 0$ and $\dot{X}^1(0) = 0$, one has $X^1(t) = 0$ for all t. ☐

12 Normal Coordinates and comparison geometry

Let us write the metric in normal coordinates: $g = g_{ij}dx^i dx^j$ Since the differential of the exponential at the origin is the identity, it follows that

$$g_{ij}(0) = \delta_{ij}.$$

In these coordinates, the straight rays from the origin are geodesics: $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0$. From this equation used at the origin, it follows that for all i, j $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}(0) = 0$ and therefore all the Christoffel symbols vanish at the origin:

$$\Gamma^k_{ij}(0) = 0.$$

Note 12.1. This is a very nice property that simplifies drastically general computations. Choosing normal coordinates to compute a formula, one can always assume that the connection terms vanish!

Caution: this is only true at one point, the derivatives of the Christoffel symbols (e.g. the curvature) do not vanish!

Finally, using the vanishing of the Christoffel symbols, it follows that the first derivatives of g_{ij} vanish at the origin, hence

$$g_{ij} = \delta_{ij} + O(r^2).$$

This means that in normal coordinates, the metric is approximated up to **second order** by the Euclidean metric $\sum(dx^i)^2$.

Remark 12.2. It is not possible in general to obtain a better approximation because the second derivatives of the coefficients g_{ij} can be interpreted as curvatures of the metric.

12.1 Metric in Geodesic Coordinates Expanded to Second Order

In terms of the Christoffel symbols Γ^k_{ij}, the components of the Riemann curvature tensor in local coordinates are given by:

$$R^l{}_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^l_{jm}\Gamma^m_{ik} - \Gamma^l_{km}\Gamma^m_{ij}.$$

This expression is derived by computing the covariant derivatives and evaluating their commutator in the coordinate basis. Note that for vectors X, Y, Z, we have in coordinates:

$$(R(X,Y)Z)^l = R^l{}_{ijk} Z^i X^j Y^k.$$

In geodesic coordinates centered at a point p on a Riemannian manifold (M, g), the metric $g_{ij}(x)$ can be expanded in powers of the radial coordinate r, which measures the distance from p, as follows:

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{iklj}(p) x^k x^l + O(r^3),$$

where:

- δ_{ij} is the Euclidean metric (the identity matrix in these coordinates),
- $R_{ikjl}(p)$ are the components of the Riemann curvature tensor at the point p,
- x^k and x^l are the coordinates in the geodesic normal coordinates centered at p,
- $O(r^3)$ denotes terms of cubic and higher orders in the radial distance r.

Example 12.3.

- The round metric in normal coordinates, is

$$g = dr^2 + \sin^2 r g_{S^{n-1}}.$$

  The injectivity radius is π.

- Similarly, the hyperbolic metric can be written in normal coordinates as

$$g = dr^2 + \sinh^2 r g_{S^{n-1}}.$$

  The injectivity radius is $+\infty$.

12.2 Gauss Lemma

Lemma 12.4 (Gauss's Lemma)**.** For any tangent vector $v \in T_pM$, let $\gamma(t)$ be the geodesic such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. If $Y \in T_pM$ is a vector **orthogonal** to v, then the vector $Y(t) \in T_{\gamma(t)}M$ defined by parallel transport of Y along $\gamma(t)$ **remains orthogonal** to $\dot\gamma(t)$ for all t. That is, for all t,

$$\langle \dot\gamma(t), Y(t)\rangle = 0.$$

Equivalently, in normal coordinates centered at p, the **radial geodesics** are orthogonal to the **distance spheres** centered at p.

This lemma has several important consequences.

1. In normal coordinates, the rays from the origin are orthogonal to the concentric spheres, which implies that

$$g = dr^2 + g_r, \quad g_r = r^2 g_{S^{n-1}} + O(r^4),$$

   with g_r a family of metrics on the sphere S^{n-1}.

2. On a ball $B \subset M$ on which exp is a diffeomorphism, the shortest path from 0 to $x \in B$ is the geodesic from 0 to x, and it is unique: geodesics are locally minimizing.

3. Small balls are convex: for any x, for r small enough, any two points of the ball $B(x, r)$ are joined by a geodesic, which is the unique shortest path between them.

12.3 The Metric g_r in Terms of Jacobi Fields

Let $S_r = \{q \in M \mid d(p, q) = r\}$ be the geodesic sphere of radius r centered at p, and let $\gamma(t)$ be a geodesic starting at p, with $\dot\gamma(0) = v \in T_pM$. The Jacobi fields $J_i(t)$ describe the separation of geodesics orthogonal to $\gamma(t)$, and they play a central role in describing the induced metric on the geodesic spheres.

Let $\{e_i(0)\}_{i=1}^n$ be an orthonormal basis of the tangent space to M^n at p so that $e_n(0) = v = \dot\gamma_0$ the initial unit speed of a geodesic γ starting at p. Let $J_i(t)$ be the Jacobi fields along γ with initial conditions

$$J_i(0) = 0, \quad \nabla_{\dot\gamma} J_i(0) = e_i(0),$$

where each $J_i(t)$ is a vector field along γ that is initially tangent to the geodesic sphere. It remains so by Gauss Lemma.

The metric g_r on the geodesic sphere S_r at radius r can be written as the inner product of the Jacobi fields $J_i(r)$. In coordinates, the induced metric g_r on S_r is given by

$$g_r = \sum_{i,j=1}^{n-1} \langle J_i(r), J_j(r)\rangle \, dy^i \otimes dy^j,$$

where dy^i are coordinate 1-forms on the geodesic sphere, and $\langle J_i(r), J_j(r)\rangle$ denotes the inner product of the Jacobi fields at $t = r$.

12.3.1 Approximation for Small r

For small values of r, the Jacobi fields approximate linear growth, and we have

$$(g_r)_{ij} = r^2\delta_{ij} + O(r^4),$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. This reflects that, at small distances, the geodesic sphere behaves like a Euclidean sphere of radius r, with curvature corrections appearing at order $O(r^4)$.

Remark 12.5. Since the sectional curvatures control the growth of the J_i, they control the growth of g_r as well. This is the basic remark of comparison geometry.

12.4 Basics of Comparison Geometry

Comparison geometry allows us to make precise geometric statements about a manifold by comparing it to simpler "model" spaces where the curvature is constant. This field of geometry relies on curvature bounds to infer properties about the geometry of a manifold, such as distances, volumes, and angles.

Here are some classical results in comparison geometry:

- **Distance comparison**: If the sectional curvature $K \geq 0$, the distance between points is controlled from below by the distance in Euclidean space. On the other hand, if $K \leq 0$, the distance between points is controlled from above by that in hyperbolic space.
- **Bonnet-Myers theorem**: If $\mathrm{Ric} \geq (n-1)k$ for some $k > 0$, then the diameter of the manifold is bounded above by the diameter of a sphere of constant curvature k.
- **Volume comparison**: If $\mathrm{Ric} \geq (n-1)k$, the volume of geodesic balls is bounded above by the volume of balls in a space of constant curvature k. This result leads to volume growth estimates for manifolds with non-negative Ricci curvature.
- **Triangle angle comparison**: For sectional curvature bounds, triangle comparison theorems imply that for $K \geq k > 0$, the angles of triangles are larger than those in a space of constant curvature k. Conversely, for $K \leq k < 0$, the angles are smaller.
- **Cartan-Hadamard theorem**: If the sectional curvature $K \leq 0$, the exponential map is globally defined, meaning that geodesics extend infinitely without self-intersections. This implies that a universal cover of a connected complete Riemannian manifold of nonpositive sectional curvature is diffeomorphic to $\mathbb{R}^n$.

12.5 Proof Sketch of the Bonnet-Myers Theorem

To give a flavor of comparison geometry, let us outline the proof of the **Bonnet-Myers theorem**, which shows that a complete Riemannian manifold with Ricci curvature bounded below by a positive constant has finite diameter.

Theorem 12.6 (Bonnet-Myers)**.** Let (M^n, g) be a complete Riemannian manifold of dimension n, and suppose $\mathrm{Ric}(X, X) \geq (n-1)k$ (implied by Sectional curvatures larger than k) for all unit tangent vectors X and some constant $k > 0$. Then the diameter of M is bounded by

$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{k}}.$$

Proof sketch if sectional curvatures bounded below by k.

1. Jacobi fields and conjugate points: Consider a geodesic $\gamma(t)$ on M. Jacobi fields along γ measure the rate of separation of nearby geodesics. A **conjugate point** occurs when a Jacobi field vanishes at two points, implying that the geodesic ceases to minimize the distance between those points.

2. Sectional curvature and Jacobi fields: The assumption $\mathrm{Sec}(g) \geq k$ implies that Jacobi fields along a geodesic γ are "more concave" than to those in a sphere of constant curvature k.

3. Bound on the length of geodesics: As a consequence, geodesics cannot minimize distances indefinitely. Specifically, if $\mathrm{Sec}(g) \geq k$, then any geodesic will encounter a conjugate point within a distance at most $\frac{\pi}{\sqrt{k}}$.

4. Conclusion: Since geodesics cannot be minimizing beyond their first conjugate point, the diameter of the manifold is at most $\frac{\pi}{\sqrt{k}}$.

□

Note 12.7. A very similar proof gives (the infinitesimal version of) Cartan-Hadamard theorem: if the sectional curvatures are negative, then Jacobi fields vanish at most once along geodesics.

13 Ricci Curvature and Scalar Curvature

Intuition 13.1. Ricci curvature and scalar curvature are ways to condense the information contained in the Riemann curvature tensor. They play crucial roles in understanding the geometric and topological properties of a manifold, with applications in general relativity and geometric analysis.

13.1 Ricci Curvature

Definition 13.2. The **Ricci curvature tensor** Ric is a symmetric $(0, 2)$-tensor obtained by tracing the Riemann curvature tensor R. In local coordinates (x^i), it is defined as:

$$\mathrm{Ric}_{ij} = R_{ik}{}^{k}{}_{j} = R^{k}{}_{ijk},$$

where R^k_{lij} are the components of the Riemann curvature tensor.

Note 13.3. The Ricci tensor measures the degree to which the volume of a geodesic ball in a manifold deviates from that in Euclidean space.

Example 13.4.

- In **Euclidean space** $\mathbb{R}^n$, the Ricci curvature vanishes identically: $\mathrm{Ric} = 0$.
- On a **sphere** $\mathbb{S}^n$ of radius r, the Ricci curvature is $\mathrm{Ric} = (n-1)\frac{1}{r^2}g$.

13.2 Scalar Curvature

Definition 13.5. The **scalar curvature** S is a scalar function obtained by taking the trace of the Ricci tensor:

$$S = g^{ij}\,\mathrm{Ric}_{ij},$$

where g^{ij} are the components of the inverse metric tensor.

Note 13.6. The scalar curvature provides a single value at each point summarizing how the volume of a small geodesic ball differs from that in Euclidean space to the second order.

Example 13.7.

- In $\mathbb{R}^n$, the scalar curvature is zero: $S = 0$.
- On $\mathbb{S}^n$ of radius r, the scalar curvature is $S = n(n-1)\frac{1}{r^2}$.

13.3 Einstein Metrics

Definition 13.8. A Riemannian metric g on a manifold M is called an **Einstein metric** if its Ricci tensor is proportional to the metric tensor:

$$\mathrm{Ric} = \lambda g,$$

for some constant λ.

Note 13.9. Einstein metrics generalize spaces of constant sectional curvature and are solutions to the vacuum Einstein field equations in general relativity.

Example 13.10.

- The standard metric on $\mathbb{S}^n$ is an Einstein metric with $\lambda > 0$.
- The flat metric on $\mathbb{R}^n$ is an Einstein metric with $\lambda = 0$.
- Hyperbolic space $\mathbb{H}^n$ with constant negative curvature is an Einstein manifold with $\lambda < 0$.

13.4 Ricci Flow

Intuition 13.11. The Ricci flow is a process that deforms the metric of a Riemannian manifold in a way that "smooths out" irregularities in its shape, much like heat diffusion smooths out temperature variations.

Definition 13.12. Given a Riemannian manifold (M, g_0), the **Ricci flow** is the evolution of the metric $g(t)$ according to the partial differential equation:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2 \operatorname{Ric}_{ij}(t),$$

with the initial condition $g(0) = g_0$.

Note 13.13. The Ricci flow was introduced by Richard S. Hamilton and has been instrumental in understanding the topology and geometry of manifolds.

- **Uniformization of Surfaces**: For two-dimensional manifolds, the Ricci flow can be used to conformally deform any metric to one of constant curvature, leading to the Uniformization Theorem.
- **Geometrization Conjecture**: In three dimensions, Ricci flow is the central tool in the proof of the Geometrization Conjecture, which classifies three-dimensional manifolds based on their geometric structures.

13.5 Curvature in Dimension 2 and 3

Intuition 13.14. In low dimensions, the Riemann and Ricci curvature tensors are significantly simpler than in higher dimensions. In two dimensions, the Gauss curvature determines the entire Riemann curvature tensor. In dimension 3, the Ricci curvature tensor becomes the central object.

In higher dimensions, all notions of curvature are very different from each other.

13.5.1 Curvature in Dimension 2

In dimension 2, the Riemann curvature tensor R_{ijkl} has only one independent component, which can be expressed entirely in terms of the **scalar curvature** S. The scalar curvature, or Gauss curvature K, therefore determines the geometry:

$$R_{ijkl} = -\frac{S}{2}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Example 13.15. For a 2-dimensional surface, Gauss curvature $K = \frac{S}{2}$ determines all other curvatures.

13.5.2 Curvature in Dimension 3

In dimension 3, it is the **Ricci curvature tensor** Ric that encodes the complete curvature information. The Riemann curvature tensor can be fully recovered from Ric as:

$$R_{ijkl} = g_{ik} \operatorname{Ric}_{jl} - g_{il} \operatorname{Ric}_{jk} + g_{jl} \operatorname{Ric}_{ik} - g_{jk} \operatorname{Ric}_{il} - \frac{S}{2}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Example 13.16. In dimension 3, Einstein metrics have constant sectional curvatures.

13.6 Proof of Gauss equation

Gauss Equation

For tangent vectors X, Y, Z, W to $M \subset \bar{M}$, where a Riemannian structure is induced on M by a Riemannian structure $(\bar{M}, \bar{g})$, one has:

$$\bar{R}(X, Y, Z, W) = \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W) \rangle - \langle \mathrm{II}(X, W), \mathrm{II}(Y, Z) \rangle + R(X, Y, Z, W).$$

Proof.
By definition:

$$\bar{R}(X,Y,Z,W) = \left\langle \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, W \right\rangle$$

The proof consists in using in the following identities:

1. $\bar{\nabla}_X Z = \left(\bar{\nabla}_X Z\right)^\top + \left(\bar{\nabla}_X Z\right)^\perp = \nabla_X Z + \mathrm{II}(X,Z)$
2. $\left\langle \bar{\nabla}_Y(\mathrm{II}(X,Z)), W \right\rangle = -\left\langle \mathrm{II}(X,Z), \bar{\nabla}_Y W \right\rangle = -\langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle$, and
3. $\left\langle \bar{\nabla}_{[Y,X]} Z, W \right\rangle = \left\langle \nabla_{[Y,X]} Z, W \right\rangle$

since for any X, Z, W tangent to M, $\mathrm{II}(X,Z)$ is orthogonal to W.

13.7 Some "Riemannian" general relativity

Intuition 13.17. In general relativity, an **initial data set** can be thought of a time slice (freeze time!) inside a space-time: some spatial 3-dimensional set from which time evolves and "reconstructs" space-time according to the (hyperbolic in Lorentzian geometry, elliptic in Riemannian geometry) Einstein equations.

Intuition 13.18. Hypersurfaces in Einstein metrics satisfy very specific coupled equations involving the first and second fundamental form. These are known as the **constraint equations**.

They are direct consequences of equations of Gauss and Codazzi.

Note 13.19. Einstein metrics in Riemannian geometry are the analogue of vacuum space-times with cosmological constant in Lorentzian geometry.

Let (M^{n+1}, g) be an Einstein manifold satisfying $\mathrm{Ric}(g) = \Lambda g$, and let $\Sigma^n \subset M$ be an embedded hypersurface with induced metric γ, second fundamental form A of a hypersurface ($\mathrm{II}(X,Y) = A(X,Y)\vec{n}$), and mean curvature $H = \mathrm{Tr}(A)$. Denote N a unit normal vector field to $\Sigma^n \subset M$ and i, j, k will denote tangential directions below.

Gauss equation: $R^\Sigma_{ijkl} = R^M_{ijkl} + A_{ik}A_{jl} - A_{il}A_{jk}$, where R^Σ is the Riemann curvature tensor of Σ, and h_{ij} is the second fundamental form.

Codazzi equation: $\nabla_i A_{jk} - \nabla_j A_{ik} = R^M_{ijkN}$, where R^M_{ijkN} is the normal-tangential component of the Riemann tensor of M.

Tracing these above equations leads to the (Riemannian version of the) constraint equations in General relativity.
The **Hamiltonian constraint**: tracing the Gauss equation and using $\mathrm{Ric}^M(g) = \Lambda g$ hence $S(g) = n\Lambda$ yields

$$S^\Sigma = (n-1)\Lambda + H^2 - |A|^2,$$

where S^Σ is the scalar curvature of Σ, H is the mean curvature, and $|A|^2 = A_{ij}A^{ij}$. Note that we also have: $R^\Sigma_{ijkl} = R^M_{ijkl} + A_{ik}A_{jl} - A_{il}A_{jk}$,
The **momentum constraint**: tracing the Codazzi equation and $\mathrm{Ric}^M_{iN} = 0$ (in i, k) yields

$$\nabla^i A_{ij} - \nabla_j H = 0,$$

which ensures the compatibility of the extrinsic curvature with the geometry of Σ.

14 Tensors, Symmetric Products, and Wedge Products

Tensors are fundamental objects in differential geometry and play a key role in the study of both local and global properties of smooth manifolds.

Intuition 14.1. A $(0,k)$ or k-tensor on a manifold M is the data at each point p of a k-linear form $(T_pM)^k \to \mathbb{R}$, the dependence in p is smooth.

Among them, the k-**differential forms** are totally antisymmetric in their argument–just like the determinant of a family of vectors. The symmetric k-differential forms are on the contrary symmetric in their arguments.

14.1 Tensor Product

Let V and W be two vector spaces over the same field $\mathbb{K}$. The **tensor product** of V and W, denoted $V \otimes W$, is a new vector space that satisfies the following universal property:

There exists a bilinear map

$$\otimes : V \times W \to V \otimes W$$

such that for any bilinear map $B : V \times W \to Z$ to any other vector space Z, there exists a unique linear map $T : V \otimes W \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr] & V \otimes W \arrow[d] \\ & Z \end{tikzcd}$$

The elements of the tensor product space $V \otimes W$ are formal linear combinations of elementary tensors of the form $v \otimes w$, where $v \in V$ and $w \in W$. These elementary tensors satisfy the following properties for all $v, v' \in V$, $w, w' \in W$, and $a, b \in \mathbb{K}$:

- $(v + v') \otimes w = v \otimes w + v' \otimes w$,
- $v \otimes (w + w') = v \otimes w + v \otimes w'$,
- $a(v \otimes w) = (av) \otimes w = v \otimes (aw)$.

The tensor product $V \otimes W$ is the vector space generated by these elementary tensors subject to these relations.

Example 14.2 (Tensor product of two vector spaces)**.** Let $V = \mathbb{R}^2$ with basis $\{e_1, e_2\}$ and $W = \mathbb{R}^3$ with basis $\{f_1, f_2, f_3\}$. The tensor product $V \otimes W$ is a vector space with dimension $2 \times 3 = 6$, and a basis is given by the set of elementary tensors:

$$\{e_1 \otimes f_1, e_1 \otimes f_2, e_1 \otimes f_3, e_2 \otimes f_1, e_2 \otimes f_2, e_2 \otimes f_3\}.$$

An arbitrary element of $V \otimes W$ can be written as a linear combination of these elementary tensors. For example,

$$v \otimes w = 2e_1 \otimes f_1 + 3e_1 \otimes f_2 + e_2 \otimes f_3$$

is an element of $V \otimes W$.

Example 14.3 (Tensor product of linear maps)**.** Let $V = \mathbb{R}^2$, $W = \mathbb{R}^3$, and $Z = \mathbb{R}$. Consider two linear maps:

$$A : V \to Z, \quad A(v_1, v_2) = v_1 + v_2,$$

$$B : W \to Z, \quad B(w_1, w_2, w_3) = w_1 - w_2 + w_3.$$

The tensor product of these linear maps is a new map $A \otimes B : V \otimes W \to Z$ defined on elementary tensors by

$$(A \otimes B)(v \otimes w) = A(v) \cdot B(w),$$

and extended by linearity to linear combinations of elements of the form $v \otimes w$. For example, if $v = (1, 2) \in V$ and $w = (1, 0, -1) \in W$, then

$$(A \otimes B)((1, 2) \otimes (1, 0, -1)) = (1 + 2) \cdot (1 - 0 - 1) = 3 \cdot 0 = 0.$$

14.2 Tensors

A **tensor** on a vector space V is a multilinear map from several copies of V and its dual V^* to $\mathbb{R}$. More formally, a (k,l)-tensor is a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \times \underbrace{V \times \cdots \times V}_{l \text{ times}} \to \mathbb{R}.$$

In terms of a basis $\{e_i\}$ of V and its dual basis $\{e^i\}$ of V^*, a tensor can be expressed as a sum of tensor products:

$$T = \sum T^{i_1 \dots i_k}_{j_1 \dots j_l}\, e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l},$$

where $T^{i_1 \dots i_k}_{j_1 \dots j_l}$ are the components of the tensor in this basis.

We will soon apply all of this to $V = (T_pM)^*$ to define the elements of $T^*M^{\otimes k}$. An element of $T^*M^{\otimes k}$ is called a $(0,k)$**-tensor**, and sometimes k**-tensor** when there is no risk of confusion.

14.3 Symmetric Products

Given a vector space V, the **symmetric product** $\mathrm{Sym}^k(V)$ is the quotient of the tensor product space $V^{\otimes k}$ by the subspace generated by tensors that are antisymmetric in any pair of arguments. In other words, a symmetric tensor is one that remains unchanged under any permutation of its arguments. Given two vectors v and w, their symmetric product is

$$v \odot w = \frac{1}{2}(v \otimes w + w \otimes v)$$

For instance, for vectors $v_1, v_2, \dots, v_k \in V$, the symmetric product is

$$v_1 \odot v_2 \odot \cdots \odot v_k = \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)},$$

where S_k denotes the symmetric group on k elements.

The space of symmetric k-tensors on V, denoted $\mathrm{Sym}^k(V)$, consists of all symmetric k-tensors, and forms a subspace of $V^{\otimes k}$.

14.4 Wedge Products

The **wedge product** (or exterior product) is an operation on the tensor algebra that produces completely antisymmetric tensors. Given two vectors $v, w \in V$, their wedge product is defined as

$$v \wedge w = \frac{1}{2}(v \otimes w - w \otimes v).$$

This operation extends to higher-degree tensors, so for vectors $v_1, v_2, \dots, v_k \in V$, their wedge product is

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)},$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ.

All of these operations can be explicited in coordinates

$$\begin{aligned}
(\alpha \wedge \beta)_{ij} &:= \frac{1}{2}(\alpha_i \beta_j - \alpha_j \beta_i), \\
(\alpha \wedge \phi)_{ijk} &:= \frac{1}{3}(\alpha_i \phi_{jk} + \alpha_j \phi_{ki} + \alpha_k \phi_{ij}), \\
(\phi \wedge \psi)_{ijkl} &:= \frac{1}{6}(\phi_{ij}\psi_{kl} + \phi_{kl}\psi_{ij} + \phi_{jk}\psi_{il} + \phi_{il}\psi_{jk} - \phi_{ik}\psi_{jl} - \phi_{jl}\psi_{ik}).
\end{aligned}$$

Definition 14.4. The space of antisymmetric k-tensors on V is denoted $\Lambda^k(V)$, also known as the space of **exterior forms** or **differential forms**.

Note 14.5. There are **many possible conventions** (sign, presence of $1/k!$ in the projection, and more) for differential forms, their norms and how they are acted on by various operators.

It is a nightmare to keep track of them in the articles you read. My advice is to always keep a set of simple examples (Euclidean space, sphere,…) on which to compare your conventions to those of articles you use as a reference.

Most of the computations you will ever have to do are formal and the dependence on convention should mostly be about computing the norms of tensors. The exact constants in front of the norms are often irrelevant to the proof.

14.5 Symmetric Tensors and Differential Forms

In differential geometry, symmetric tensors and differential forms are used to describe various geometric structures on manifolds.

- A **symmetric tensor field** of rank k on a manifold M assigns a symmetric k-tensor to each point $p \in M$. The space of symmetric k-tensors at p is denoted $\mathrm{Sym}^k(T_p^*M)$, where $T_p^*M = (T_pM)^*$ is the cotangent space at p.
- A **differential form** of degree k on a smooth manifold M is an antisymmetric k-tensor field. The space of all differential k-forms on M is denoted $\Omega^k(M)$, where $\Omega^k(M) = \Lambda^k(T^*M)$, the exterior power of the cotangent bundle.

15 Differential Forms, Stokes' theorem and de Rham Cohomology

Differential forms and de Rham cohomology provide essential tools for studying the topology of smooth manifolds. By using differential forms, we can construct cohomology groups that classify certain types of topological information about a manifold.

This is a general theme at the intersection of geometry, analysis and topology: counting the number of solutions of specific PDEs lets one understand the topology of the manifold.

15.1 Differential Forms in coordinates

A **differential k-form** on a smooth manifold M is a smooth, antisymmetric, multilinear map that assigns to each point $p \in M$ an antisymmetric k-linear form on the tangent space T_pM. Specifically, a k-form ω can be expressed locally in coordinates $(x^1, \dots, x^n)$ as

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

where $\omega_{i_1 \dots i_k}$ are smooth functions on M, and $dx^{i_1} \wedge \dots \wedge dx^{i_k}$ are the **wedge products** of the coordinate 1-forms dx^i.

Example 15.1.

- A 0-form is a smooth function.
- A 1-form is an element of T^*M: a linear form on each T_pM.
- Any n-form ω on M^n is of the product of a smooth function times the volume form of a given Riemannian metric g: $\omega = f \, dv_g$. In particular, for any other Riemannian metric g', one sometimes denotes abusively $\frac{dv_{g'}}{dv_g}$ the function so that $dv_{g'} = \frac{dv_{g'}}{dv_g} \, dv_g$
- If there is a repeated index $i_a = i_b$ for $a \neq b$ in $dx^{i_1} \wedge \dots \wedge dx^{i_k}$, then $dx^{i_1} \wedge \dots \wedge dx^{i_k} = 0$. As a consequence, on an n-dimensional manifold, any k-form for $k \geqslant n + 1$ vanishes.

Note 15.2. An n-manifold is called **orientable** if there exists a global *nonvanishing* n-form.

15.2 The Exterior Derivative

The **exterior derivative** d is an operator that takes a k-form $\omega \in \Omega^k(M)$ and produces a $(k+1)$-form $d\omega \in \Omega^{k+1}(M)$. The exterior derivative has the following properties:

1. **Linearity**: $d(a\omega + b\eta) = a \, d\omega + b \, d\eta$ for any forms ω, η and scalars a, b,
2. **Leibniz rule**: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,
3. **Nilpotence**: $d(d\omega) = 0$ for any $\omega \in \Omega^k(M)$.

Definition 15.3 (Closed and exact forms)**.** An **exact** form is a form that can be written as $d\eta$ for some η.

A **closed** form is a form ω such that $d\omega = 0$.

The nilpotence property $d \circ d = 0$ implies that any **exact** form is also **closed**.

In local coordinates, the exterior derivative of $\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, is

$$d\omega = \sum_{i_1<\cdots<i_k} d(\omega_{i_1\ldots i_k}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where, once again in coordinates: for any function f, one has

$$d(f) = \sum_{l=1}^{n} (\partial_{x^i} f)\, dx^i.$$

15.3 de Rham Cohomology

The **de Rham cohomology groups** $H^k_{\mathrm{dR}}(M)$ of a smooth manifold M provide a way to classify differential forms based on their closedness and exactness properties. The k-th de Rham cohomology group is defined as:

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

In other words:

- The **kernel** $\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))$ consists of all **closed** k-forms, those k-forms ω for which $d\omega = 0$.
- The **image** $\mathrm{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))$ consists of all **exact** k-forms, those that can be written as $d\eta$ for some $(k-1)$-form η.

Intuition 15.4. These groups measure the "holes" or non-trivial cycles in M that cannot be "filled in".

The de Rham cohomology group $H^k_{\mathrm{dR}}(M)$ captures the equivalence classes of closed k-forms modulo exact k-forms.

Definition 15.5. The de Rham **cohomology class** of a closed k-form ω is

$$[\omega] = \{\omega' = \omega + d\eta \in \Omega^k(M), \eta \in \Omega^{k-1}\}.$$

15.4 Properties of de Rham Cohomology

De Rham cohomology has several important properties:

1. **Homotopy invariance**: If M and N are homotopy equivalent (in particular diffeomorphic), then $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$.
2. **Poincaré duality**: For a closed, oriented n-dimensional manifold M, there is an isomorphism $H^k_{\mathrm{dR}}(M) \cong H^{n-k}_{\mathrm{dR}}(M)$ induced by the wedge product and integration.
3. **Mayer-Vietoris sequence**: There is a long exact sequence relating the cohomology of a manifold M to the cohomology of two open sets covering M. This is useful for computing cohomology in terms of simpler pieces.

15.5 Example: de Rham Cohomology of the Circle

Example 15.6. Consider $M = S^1 = [0,1]/\approx$, the unit circle with $0 \approx 1$ parametrized by x. We compute the de Rham cohomology groups of S^1:

- $H^0_{\mathrm{dR}}(S^1)$: The 0-forms on S^1 are simply smooth functions, and the closed 0-forms are constant functions. Thus, $H^0_{\mathrm{dR}}(S^1) \cong \mathbb{R}$.
- $H^1_{\mathrm{dR}}(S^1)$: The 1-forms on S^1 that are closed (with $d\omega = 0$) but not exact (cannot be written as df for a function f) are spanned by dx. Indeed, dx is a valid 1-form on S^1, but x is not a smooth (or continuous) function on S^1 since $x(0) \neq x(1)$.

  They capture the non-trivial loop structure of S^1. We find $H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$.

Note 15.7. Higher cohomology groups are zero for S^1, so $H^k_{\mathrm{dR}}(S^1) = 0$ for $k \geq 2$.

15.6 Stokes' Theorem

Stokes' theorem is a powerful result in differential geometry that generalizes several classical theorems from vector calculus, such as the Fundamental Theorem of Calculus, Green's Theorem, and the Divergence Theorem. This is the central theorem for integration by parts.

Theorem 15.8 (Stokes' Theorem)**.** Let M be an oriented, smooth manifold with boundary ∂M, and let ω be a differential $(k-1)$-form on M. Then, Stokes' theorem states:

$$\int_M d\omega = \int_{\partial M} \omega.$$

In particular, for a closed (without boundary) manifold M, $\int_M d\omega = 0$.

Example 15.9.

- **Green's Theorem**: For a region $R \subset \mathbb{R}^2$ with boundary ∂R, Green's theorem is a special case of Stokes' theorem applied to a 1-form in the plane. It states:

  $$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial R} P\,dx + Q\,dy,$$

  where P and Q are the components of a vector field on $\mathbb{R}^2$.
- **The Divergence Theorem**: In $\mathbb{R}^3$, the Divergence Theorem relates the integral of the divergence of a vector field over a region V to the flux of the vector field through the boundary ∂V:

  $$\int_V \nabla \cdot \mathbf{F}\,dV = \int_{\partial V} \mathbf{F} \cdot \mathbf{n}\,dS,$$

  where $\mathbf{F}$ is a vector field, $\mathbf{n}$ is the outward-pointing unit normal to the surface ∂V, and dS is the surface element.
- **Fundamental Theorem of Calculus**: In the case of a 1-dimensional manifold $M = [a, b] \subset \mathbb{R}$, Stokes' theorem reduces to the Fundamental Theorem of Calculus:

  $$\int_a^b f'(x)\,dx = \int_a^b d(f) = f(b) - f(a),$$

  where f is a smooth function on $[a, b]$.

We will see many applications of this theorem: this is our tool to **integrate by parts on manifolds.**

15.7 From submanifolds to closed differential forms

Let $X^k \subset M^n$ be a submanifold of dimension k, and let $N \to X$ be the normal bundle of X.

15.7.1 Tubular neighborhoods and normal exponential

A **tubular neighborhood** $\nu(X)$, that is a (small) open set in M^n so that $X \subset \nu(X) \subset M^n$. The most common way to obtain such a neighborhood is to consider an ambient metric g on M. To construct a tubular neighborhood, we can use the normal exponential map associated with the ambient metric g.

Definition 15.10. Let $N \to X$ be the **normal bundle** of X in M, which consists of the vectors orthogonal to X with respect to the metric g. For each point $p \in X$, the fiber of N at p, denoted N_p, is the orthogonal complement of the tangent space T_pX in T_pM, i.e.,

$$N_p = \{v \in T_pM \mid g(v, w) = 0 \text{ for all } w \in T_pX\}.$$

Definition 15.11. The **normal exponential map** is defined as

$$\exp^\perp : N \to M, \quad \exp^\perp(v) = \exp_p(v),$$

where $\exp_p$ is the Riemannian exponential map at $p \in X$ in M, and $v \in N_p$ is a normal vector to X at p. For sufficiently small vectors v, the map $\exp^\perp(v)$ yields a point in M close to X.

In other words, for each point $p \in X$, the normal exponential map sends normal vectors $v \in N_p$ to points in M by following the geodesics emanating from p in the directions of v.

Note 15.12. Many of the properties of the standard normal coordinates based at a *point*, that is a 0-dimensional submanifold. In particular, the normal exponential map to X is a diffeomorphism up until a distance called the **normal injectivity radius**.

If both M and X are compact, a tubular neighborhood of X is realized as the image of a small neighborhood around the zero section of the normal bundle N under the normal exponential map. Specifically, for sufficiently small $\epsilon > 0$, the set

$$\nu(X) = \exp^\perp(\{v \in N \mid \|v\| < \epsilon\})$$

forms a tubular neighborhood of X in M.

15.7.2 Thom's class

We will construct the **Thom form** $\omega \in \Omega^{n-k}(M)$, which is supported in $\nu(X)$. As seen above, we may choose $\nu(X)$ as diffeomorphic to $\{v \in N \mid \|v\| < \epsilon\}$. We will construct a closed form on this set that is "dual" to X: the zero section of N. Considering a local trivialization of $N \approx U \times \mathbb{R}^{n-k} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, we will further construct a form that is closed on $U \times \mathbb{R}^{n-k}$ with natural coordinates $(x^1, \dots, x^n)$.

1. Consider r_N to be the **normal distance**: $r_N^2(x^1, \dots, x^n) = (x^{k+1})^2 + \cdots + (x^n)^2$.
2. Consider a **cut-off function** $\chi : \mathbb{R}^+ \to [0, 1]$ supported in $[0, \epsilon/2]$ and equal to 1 on $[0, \epsilon/4]$.
3. Consider the form ω defined by:
$$\omega := \chi(r_N) dx^{k+1} \wedge \cdots \wedge dx^n.$$
4. We may normalize the form ω by multiplying it by a constant factor.

Then ω is a **closed** form since $d\chi(r_N)$ is a C^∞-combination of $dx^{k+1}, \dots, dx^n$ only. The cohomology class of ω is the **Thom class** dual to the submanifold X. Note that this class can be constructed as supported on an arbitrarily small neighborhood of $X \subset M$.

16 Hodge Theory

Hodge theory is a fundamental area of differential geometry and **global analysis** that connects the geometry of a smooth, compact Riemannian manifold (M, g) with its topology. The core idea of Hodge theory is that on such manifolds, each de Rham cohomology class has a unique g-harmonic representative.

Note 16.1. None of the notions of differential form, wedge product, exterior derivative and de Rham cohomology depend on a metric and can be considered purely topological. In contrast, Hodge theory **depends on a metric**.

16.1 The Hodge Star Operator

Given a smooth, compact, oriented Riemannian n-manifold M with metric g, the **Hodge star operator** $*$ is an isomorphism that maps k-forms to $(n-k)$-forms. The operator $*_g \colon \Omega^k \to \Omega^{n-k}$ is defined by the property:

$$\alpha \wedge *_g \beta = g(\alpha, \beta)\, dv_g,$$

where α and β are k-forms, $g(\alpha, \beta)$ is the inner product of the forms induced by the Riemannian metric, and dv_g is the volume form of M, g.

Note 16.2. The convention for $g(\alpha, \beta)$ here is that for $\eta_k := e^1 \wedge \cdots \wedge e^k$ for $(e^i)_{i \in \{1,\ldots,n\}}$ a g-orthonormal basis of $(T_p M)^*$,

$$g(\eta_k, \eta_k) = 1.$$

Other (rare) conventions the Hodge star $*_g\, \alpha \wedge \beta = g(\alpha, \beta)\, dv_g$.

Note 16.3. Just like the Hodge star, the codifferential does depend on the metric. This is not the case of the exterior derivative d.

Remark 16.4. The Hodge star operator satisfies $*(*\,\omega) = (-1)^{k(n-k)}\omega$, making it an involution up to a sign.

Example 16.5. If $(e_i)_{i=1,\ldots,n}$ is a direct orthonormal basis of V, then $(e_I)_{I \subset \{1,\ldots,n\}}$ forms an orthonormal basis of ΩV. One checks easily that:

$$* 1 = \mathrm{vol}, \quad * e_1 = e_2 \wedge e_3 \wedge \cdots \wedge e_n,$$
$$* \mathrm{vol} = 1, \quad * e_i = (-1)^{i-1} e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n.$$

16.2 The Hodge Laplacian

The **Hodge Laplacian** Δ is a second-order differential operator on differential forms. For a k-form ω, it is defined as

$$\Delta\omega = (d\delta + \delta d)\omega = (dd^* + d^* d)\omega,$$

where d is the exterior derivative and

$$d^* = \delta = (-1)^{nk+n+1} * d *$$

is the **codifferential**, which maps k-forms to $(k-1)$-forms, it is the *formal adjoint* of d, see the definition below.

Note 16.6. The Hodge Laplacian Δ is self-adjoint and non-negative, making it a key operator in spectral theory on manifolds.

Note 16.7. Once again, there are two conventions: the "**analyst's Laplacian** convention" corresponding to

$$\Delta = \sum_i \partial^2_{x_i^2}$$

and as above, the "**geometer's Laplacian** convention" corresponding to the L^2-positive operator

$$\Delta = -\sum_i \partial^2_{x_i^2}.$$

Example 16.8. On functions, the Hodge Laplacian is also called the Laplace-Beltrami operator and equals $\Delta = \delta d$. The Laplace-Beltrami operator Δ for a function f on an n-dimensional Riemannian manifold with metric g is given by

$$\Delta f = -\frac{1}{\sqrt{\det(g_{ij})_{ij}}}\partial_i\left(\sqrt{\det(g_{ij})_{ij}}\, g^{ij}\, \partial_j f\right).$$

16.3 Harmonic Forms

A differential form ω is called **harmonic** if it satisfies

$$\Delta\omega = 0.$$

The space of harmonic k-forms on M is denoted by $\mathcal{H}^k(M)$. Since Δ is an elliptic operator, $\mathcal{H}^k(M)$ is finite-dimensional (see next section).

Proposition 16.9. Harmonic forms are both closed and coclosed on compact manifolds, meaning $d\omega = 0$ and $\delta\omega = 0$.

Proof. Let ω be a harmonic form. The proof is an integration by parts against ω in the present context.

We will use two formulae in combination with Stokes' theorem: using the above formulae for δ and $* \circ *$, we find:

1. $d(\delta\omega \wedge *\,\omega) = d\delta\omega \wedge *\,\omega + (-1)^{k-1}\delta\omega \wedge d *\omega = d\delta\omega \wedge *\,\omega - \delta\omega \wedge *\,\delta\omega = \left(\langle d\delta\omega, \omega\rangle - \|\delta\omega\|^2\right) dv$, and
2. $d(\omega \wedge *\, d\omega) = d\omega \wedge *\, d\omega + (-1)^k\omega \wedge d * d\omega = \left(\|d\omega\|^2 - \langle\omega, \delta d\omega\rangle\right) dv.$

□

16.4 The Hodge Theorem

The **Hodge theorem** establishes a fundamental correspondence between cohomology classes and harmonic forms. For a smooth, compact, oriented Riemannian manifold M, the theorem states:

$$H^k_{\mathrm{dR}}(M) \cong \mathcal{H}^k(M),$$

where $H^k_{\mathrm{dR}}(M)$ is the k-th de Rham cohomology class of M and $\mathcal{H}^k(M)$ is the space of harmonic k-forms on M.

More precisely, this theorem implies that each cohomology class has a unique harmonic representative, allowing us to study topological properties of M through the analysis of the PDE $\Delta\omega = 0$.

16.5 Applications and Consequences of Hodge Theory

Hodge theory has several significant applications:

1. **Topological Invariants**: The dimensions of the spaces of harmonic forms $\mathcal{H}^k(M)$, known as the **Betti numbers**, provide topological invariants that describe the shape of M.
2. **Hodge Decomposition**: Hodge theory provides a decomposition of the space of k-forms on M as

   $$\Omega^k(M) = \mathcal{H}^k(M) \oplus \mathrm{im}(d) \oplus \mathrm{im}(\delta),$$

   where $\mathrm{im}(d)$ and $\mathrm{im}(\delta)$ are the images of the exterior derivative and the codifferential, respectively.
3. **Hodge Duality**: Hodge theory establishes a duality between k-forms and $(n-k)$-forms on M via the Hodge star operator, enabling analysis of cohomology groups in terms of dual spaces.

17 Hodge theorem: a global analysis proof

Global analysis is a subfield of geometric analysis that studies the solvability and set of solutions of PDEs globally on manifolds. This solvability is often related to 1) curvature conditions satisfied on a manifold, and 2) to the topology of the manifold. It is central in geometry, physics and topology.

17.1 Differential Operators and their symbols

Intuition 17.1. Before talking about *global* analysis, differential operators often have *local* regularizing properties. This is how we can get away with studying PDEs on *smooth sections* only.

Definition 17.2. A linear operator $P : \Gamma(M, E) \to \Gamma(M, F)$, acting on sections of two vector bundles E and F over a smooth manifold M, is called a **differential operator** of order d if in any local trivialization of E and F, it can be written in coordinates as

$$Pu(x) = \sum_{|\alpha| \leq d} a_\alpha(x) \partial^\alpha u(x),$$

where α is a multi-index $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n |\alpha_i|$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, and $a_\alpha(x)$ are smooth coefficients.

Definition 17.3. The highest order term, where $|\alpha| = d$, defines the **principal symbol** $\sigma_P(x, \xi)$ of the operator, which is a homogeneous polynomial of degree d in the cotangent variable $\xi \in T_x^* M$, given by

$$\sigma_P(x, \xi) = \sum_{|\alpha| = d} a_\alpha(x) \xi^\alpha,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$ if $\xi = \xi_i dx^i$. It is a degree d homogeneous polynomial in the variable ξ with values in $\mathrm{Hom}(E_x, F_x) \approx E_x^* \otimes F_x$.

Remark 17.4. Here is another equivalent definition that justifies the intrinsic nature of the principal symbol (it does not depend on local coordinates): suppose $f \in C^\infty(M)$, $t \in \mathbb{R}$, and $u \in \Gamma(M, E)$, then

$$e^{-tf(x)} P(e^{tf(x)} u(x))$$

is a polynomial of degree d in the variable t, whose monomial of degree d is a homogeneous polynomial of degree d in $df(x)$, which is

$$t^d \sigma_P(x, df(x)) u(x).$$

Intuition 17.5. The principal symbol isolates the leading behavior of the operator P at small scale. Up to restricting our attention to very small sets (through a partition of unity), the main analytic properties of the operator are seen in the principal symbol.

This principal symbol is crucial in understanding the behavior of the operator and is intrinsic to the differential operator itself, independent of local coordinates.

Example 17.6.

- The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ has a principal symbol $\sigma_d(x, \xi) = \xi \wedge$.
- Its formal adjoint $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$ has symbol $\sigma_{d^*}(x, \xi) = \xi \lrcorner$, where $\lrcorner$ denotes the interior product: $X \lrcorner \omega = \omega(X, \cdot)$.
- The Hodge Laplacian $\Delta = dd^* + d^* d$ has principal symbol $\sigma_\Delta(x, \xi) = |\xi|^2$.
- Consider a connection $\nabla : \Gamma(E) \to \Gamma(\Omega^1 \otimes E)$, then $e^{-tf} \nabla(e^{tf} u) = t\, df \otimes u + \nabla u$. Therefore,

$$\sigma_\nabla(x, \xi) = \xi \otimes : \Gamma(E_x) \to \Gamma(\Omega^1_x \otimes E_x).$$

17.2 Elliptic Regularity

Elliptic regularity is a fundamental property of elliptic partial differential equations (PDEs) that relates the regularity (smoothness) of solutions to the regularity of the data. This result is crucial for proving that the solutions to elliptic PDEs are actually smooth tensors, provided the data is smooth.

Definition 17.7. An operator P is called **elliptic** if its principal symbol $\sigma_P(x, \xi)$ is invertible for all $x \in M$ and all nonzero $\xi \in T_x^* M$.

Elliptic operators have several important properties detailed below, including:

- **Finite-dimensional kernel and cokernel**: The space of solutions to the equation $Pu = 0$ is finite-dimensional.
- **Elliptic regularity**: Solutions of elliptic equations are smooth if the data are smooth, i.e., if $Pu = f$ and f is smooth, then u is also smooth.

Let P be a differential operator acting on sections of a vector bundle E over a smooth manifold M:

$$P : \Gamma(M, E) \to \Gamma(M, F).$$

17.2.1 Basic Elliptic Regularity Theorem

The key result of **elliptic regularity** can be stated as follows:

Theorem 17.8 (Elliptic Regularity)**.** Let P be an elliptic operator of order d on a compact manifold M, and suppose $u \in H^s(M, E)$ satisfies $Pu = f$ with $f \in H^t(M, F)$ for some $t \in \mathbb{R}$. Then, if $u \in L^2(M, E)$, we have

$$u \in H^{t+d}(M, E).$$

In other words, the solution u gains d derivatives of regularity compared to the data f.

Here $H^s(M, E)$ denotes the Sobolev space of sections of the bundle E of class H^s. If P is of order d, this result implies that if $f \in H^t$, then $u \in H^{t+d}$, meaning that the solution becomes smoother as long as the data is smooth.

A more refined form of the elliptic regularity theorem is given by **elliptic estimates**, which provide bounds on the Sobolev norms (or Hölder in different contexts) of solutions. If P is elliptic and u satisfies $Pu = f$, then for any integer $s \geq 0$, there exists a constant C such that

$$\|u\|_{H^{s+d}} \leq C \left(\|Pu\|_{H^s} + \|u\|_{L^2} \right).$$

This inequality ensures that the Sobolev norm of the solution u can be controlled by the Sobolev norm of $f = Pu$ and the L^2-norm of u itself. The L^2-norm term is necessary because elliptic operators are generally not injective, so there may be non-trivial solutions in the kernel of P, and elliptic estimates hold up to such kernel elements.

Note 17.9. All of the proofs of this section rely on Sobolev or Hölder spaces with **finite regularity** H^s and basic functional analysis in Banach spaces. We will hide all of the details and talk about "smooth" harmonic functions, their complement etc. The reason is that the above elliptic estimates together with Sobolev embeddings say that if a solution u of Pu is merely L^2, it becomes automatically C^∞.

Remark 17.10. A great exercise is to prove this kind of elliptic estimate on a circle or a torus using Fourier decomposition: any compact set of a manifold can in particular be seen as an open set of a torus.

17.3 Formal Adjoints and Basic Elliptic Theory

Intuition 17.11. The notion of (formal) adjoint generalizes the transpose of matrices in finite-dimensional linear problems.

Given a differential operator $P : \Gamma(M, E) \to \Gamma(M, F)$, its **formal adjoint** $P^* : \Gamma(M, F) \to \Gamma(M, E)$ is defined through the L^2-inner product:

$$(Pu, v)_F = (u, P^* v)_E \quad \text{for all } u \in \Gamma(M, E), v \in \Gamma(M, F).$$

For example, the formal adjoint of the exterior derivative d is the codifferential d^*, and its explicit expression in local coordinates can be derived via **integration by parts**.

17.4 Fredholm Alternative

Intuition 17.12. The cokernel measures the failure of P to be surjective; it captures those elements in $\Gamma(M, F)$ that are not in the image of P. The kernel detects the failure to being injective.

For a differential operator $P : \Gamma(M, E) \to \Gamma(M, F)$, the **cokernel** of P, denoted as $\operatorname{coker}(P)$, is defined as the quotient space

$$\operatorname{coker}(P) = \Gamma(M, F) / \operatorname{im}(P),$$

where $\operatorname{im}(P)$ is the image of P. By elliptic regularity, elliptic operators have a closed image in appropriate Sobolev spaces, so $\operatorname{coker}(P)$ is finite-dimensional for elliptic operators. To identify the cokernel of P, one typically uses the formal adjoint operator P^*.

The **Fredholm alternative** is a key result in the theory of elliptic operators (or more generally Fredholm operators).

It asserts the following for an elliptic operator $P : \Gamma(M, E) \to \Gamma(M, F)$ between sections of vector bundles over a compact manifold:

1. $\dim(\ker(P))$ is finite.
2. $\dim(\operatorname{coker}(P)) = \dim(\ker(P^*))$.
3. The image of P is closed, and there is an L^2-orthogonal decomposition

$$\Gamma(M, F) = \operatorname{im}(P) \oplus \ker(P^*).$$

Thus, every element $v \in \Gamma(M, F)$ can be decomposed uniquely as

$$v = Pu + w, \quad \text{where } u \in \Gamma(M, E) \text{ and } w \in \ker(P^*).$$

This decomposition is crucial for establishing solvability conditions for elliptic PDEs: an equation $Pu = v$ has a solution if and only if v is L^2-orthogonal to $\ker(P^*)$, i.e.,

$$(v, w)_{L^2} = 0 \quad \forall w \in \ker(P^*).$$

This is the core of the Fredholm alternative, which ensures that the operator P has a well-behaved inverse modulo its finite-dimensional kernel and cokernel.

Intuition 17.13. The Fredholm alternative identifies the cokernel of an elliptic operator as the kernel of its formal adjoint. The kernel of the formal adjoint becomes the **obstruction** to solving equations. Typically,

$$Pu = v$$

is solvable *if and only if* $v \perp_{L^2} \ker(P^*)$.

Elliptic regularity is a foundational result in global analysis, with numerous applications, including:

- **Index theory**: The study of elliptic operators is central to the Atiyah-Singer index theorem, which relates the analytical properties of elliptic operators to topological invariants of the underlying manifold. The **index**, the difference between the dimensions of the kernel of P and that of P^* is such an analytical property that detects topological invariants.
- **Regularity of solutions to geometric PDEs**: In the study of geometric partial differential equations, such as the Einstein equation, minimal surfaces or the Yamabe problem, elliptic regularity ensures that weak solutions are smooth, thereby allowing the application of geometric and topological techniques.
- **Inverse operators**: Elliptic regularity also plays a key role in proving the existence of inverse operators for elliptic operators. Inverse operators, like inverse matrices are central to solving linear equations.
- **Fredholmness of elliptic operators**: Ellipticity of the operators implies that up to **finite-dimensional and closed** kernels and cokernels, the operators are **invertible**. This in combination with Banach's inverse or open theorem often reduces solving (nonlinear) PDEs to the study its behavior between these finite-dimensional spaces.

17.5 Selfadjoint Operator

An operator $P : \Gamma(M, E) \to \Gamma(M, F)$ is said to be **selfadjoint** if $P = P^*$, meaning that for all sections u, v with compact support, we have

$$(Pu, v)_{L^2} = (u, Pv)_{L^2}.$$

In other words, P coincides with its formal adjoint, making the operator symmetric with respect to the L^2-inner product.

Selfadjoint elliptic operators have several important properties:

- **Real spectrum**: The eigenvalues of a selfadjoint operator are real.
- **Orthogonal eigenfunctions**: The eigenfunctions corresponding to distinct eigenvalues are orthogonal in the L^2-sense.
- **Decomposition of the space**: For compact selfadjoint operators, the space $\Gamma(M, E)$ can be decomposed into an orthonormal basis of eigenfunctions, and the operator acts diagonally with respect to this basis. The simplest example being Fourier series for functions on $\mathbb{S}^1$, it generalizes to k-forms on any manifold.

A key example of a selfadjoint operator is the **Hodge Laplacian**, defined on p-forms as

$$\Delta = d\delta + \delta d,$$

which is formally selfadjoint. The **Hodge theorem** follows from the fact that the Laplacian is elliptic and selfadjoint, allowing us to decompose the space of differential forms in the following L^2-*orthogonal* way: the **Fredholm alternative** applied to Δ gives

$$\Omega^p(M) = \ker(\Delta) \oplus \operatorname{im}(\Delta) = \ker(\Delta) \oplus \Delta(\Omega^p(M)).$$

Here, $\ker(\Delta)$ consists of harmonic p-forms, and the Hodge theorem asserts that every cohomology class has a unique harmonic representative, completing the proof of Hodge theory on compact manifolds.

17.6 The Operator $d + \delta$ as a Dirac-like Operator

We now refine Hodge theorem, replacing Δ by the action of d and δ alone.

The operator $d + \delta : \Omega^k \to \Omega^{k+1} \oplus \Omega^{k-1}$ can be viewed as a Dirac-like operator from differential forms of even (resp. odd) order to differential forms of odd (resp. even). **Dirac operators** are generally understood to be first-order elliptic operators that square to some kind of Laplacian. This is the case here since $d \circ d = 0$ and $\delta \circ \delta = 0$,

$$(d + \delta)^2 = d\delta + \delta d = \Delta,$$

where $\Delta = d\delta + \delta d$ is the Hodge Laplacian. The interpretation of $d + \delta$ as a Dirac-like operator allows us to apply techniques to problems involving differential forms.

Remark 17.14. The analogy with spin geometry and Dirac operators goes further, one can define a Clifford multiplication, etc. on the space of differential forms.

17.6.1 Consequences for the Hodge Theorem

Since the operator $d + \delta$ squares to the Hodge Laplacian Δ, its kernel corresponds to harmonic forms, i.e., forms that satisfy both $d\alpha = 0$ and $\delta\alpha = 0$. These harmonic forms play a central role in Hodge theory, as they provide representatives for cohomology classes.

The Hodge theorem can consequently be refined as the following L^2-**orthogonal** decomposition:

$$\Omega^p(M) = \mathcal{H}^p(M) \oplus \mathrm{im}(d) \oplus \mathrm{im}(\delta) = \mathcal{H}^p(M) \oplus d(\Omega^{p-1}(M)) \oplus \delta(\Omega^{p+1}(M)),$$

where $\mathcal{H}^p(M) = \ker(\Delta) = \ker(d + \delta)$ denotes the space of harmonic p-forms.

The operator $d + \delta$, as a Dirac-like operator, contributes to this decomposition by providing a mechanism to "link" the exact and coexact forms through its ellipticity, and by doing so, it reinforces the orthogonal decomposition of the space of differential forms.

Note 17.15. The de **Rham complex**, consisting of the exterior derivative d on differential forms,

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \to 0,$$

is a classic example of an **elliptic complex**. The ellipticity of the de Rham complex follows from the ellipticity of the Hodge Laplacian $\Delta = d\delta + \delta d$, as discussed earlier.

17.6.2 Consequences for Hodge Theory

In the context of the Hodge theorem, the de Rham complex being elliptic has several key consequences:

- **Finite-dimensional cohomology**: Since the de Rham complex is elliptic, the cohomology groups $H^p(M) = \ker(d_{|\Omega^p}) / \mathrm{im}(d_{|\Omega^{p-1}})$ are finite-dimensional on a compact manifold.
- **Harmonic representatives**: Elliptic regularity ensures that solutions to the Laplace equation $\Delta\alpha = 0$ are smooth, so each cohomology class has a smooth harmonic representative.

17.6.3 Construction of harmonic representative

Let us explain how to find the g-harmonic representative of a de Rham cohomology $[\omega]$ for some closed k-form ω. One could for instance start with $[\omega]$ the Thom's class of some submanifold.

We only know that $d\omega = 0$ at first. We are looking for $\alpha \in \Omega^{k-1}$ so that

$$\delta(\omega + d\alpha) = 0,$$

since $d(\omega + d\alpha) = 0$ is automatic by nilpotence.

This rewrites as the following PDE with unknown α:

$$\delta d\alpha = -\delta\omega$$

Now, since $\alpha \in \Omega^{k-1} = \mathcal{H}^{k-1} \oplus d(\Omega^{k-2}) \oplus \delta(\Omega^k)$, we may assume that $\alpha \in \delta(\Omega^k)$ since the rest is killed by d. Under this assumption

$$\delta d\alpha = \Delta\alpha = -\delta\omega$$

becomes an elliptic equation. The Fredholm alternative tells us that it is solvable if and only if $-\delta\omega \perp_{L^2} \ker\Delta = \ker d \cap \ker\delta$. This is the case by integration by parts: let $\beta \in \ker\Delta$, then since $d\beta = 0$,

$$\int_M \langle \delta\omega, \beta\rangle dv = \int_M \langle \omega, d\beta\rangle dv = 0.$$

Finally, one verifies that the resulting $\alpha \in \delta(\Omega^k)$ as assumed.

Note 17.16. Assuming that $\alpha \in \delta(\Omega^k)$, or that $\delta\alpha = 0$ is a common "gauge fixing" assumption. It correspond to the divergence-free, or Bianchi free condition for the Einstein equation or in fluid dynamics, or to the Coulomb gauge in Yang-Mills theory.

17.7 Maxwell's Equations as a Harmonic Equation

Maxwell's equations, which govern electromagnetism, can be described using differential forms, linking the theory to the Hodge Laplacian and elliptic theory. In vacuum, in order to state Maxwell's equations, consider the electric field E, and the magnetic field B, seen as a *time-dependent* 1-form on $\mathbb{R}^3$. Denote $*_3$, the 3-dimensional (in the spatial direction) Hodge star.

To express these equations in the language of differential forms, we introduce the **Faraday 2-form** F on spacetime $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ with coordinates (t, x, y, z), which combines the electric and magnetic fields. This 2-form is given by

$$F = dt \wedge E + *_3\, B.$$

Definition 17.17. The **Maxwell equations in vacuum** can be written in terms of the exterior derivative as

$$dF = 0, \quad d^*F = 0,$$

where $\star$ denotes the Hodge star operator associated with the metric on M. These two equations state that F is a *closed* and *coclosed* 2-form, or equivalently, that F is a *harmonic* 2-form on M with respect to the Hodge Laplacian $\Delta = dd^* + d^*d$.

Since F is both closed and coclosed, it lies in the kernel of the Hodge Laplacian:

$$\Delta F = 0.$$

The Hodge theorem then guarantees that, on a compact manifold, any cohomology class has a unique harmonic representative, which here implies that physical electromagnetic fields can be identified with harmonic forms in the appropriate cohomology class of M.

Remark 17.18. This perspective provides a natural link between electromagnetism and topology, as the harmonic forms corresponding to Maxwell's equations are elements of the de Rham cohomology groups $H^2(M)$. In particular, the topology of the underlying manifold M may impose constraints on the existence of solutions, depending on the cohomological properties of M.

Note 17.19. Maxwell's equations are the Euler-Lagrange equations associated to the "energy" $F \mapsto \int_M |F|_g^2 dv_g$ when F is restricted to a given cohomology class.

18 Bochner technique and Weitzenböck formulae

18.1 Action of the Levi-Civita Connection on (p, q)-Tensors

For a (p, q)-tensor field T on a Riemannian manifold M, the Levi-Civita connection provides a means of computing directional derivatives that generalize the concept of partial derivatives in Euclidean space.

18.1.1 Definition of the Levi-Civita Connection on (p, q)-Tensors

For a (p, q)-tensor T, which can be represented locally by components $T^{a_1\ldots a_p}_{b_1\ldots b_q}$, the Levi-Civita connection acts on T by differentiating each of its components while preserving the index structure. Explicitly, the covariant derivative of T in the direction of a vector field X is given by:

$$(\nabla_X T)^{a_1\ldots a_p}_{b_1\ldots b_q} = X(T^{a_1\ldots a_p}_{b_1\ldots b_q}) + \sum_{i=1}^{p} T^{a_1\ldots c\ldots a_p}_{b_1\ldots b_q}\,\Gamma^{a_i}_{c,k}X^k - \sum_{j=1}^{q} T^{a_1\ldots a_p}_{b_1\ldots c\ldots b_q}\,\Gamma^{c}_{b_j,k}X^k,$$

where Γ^a_{bc} are the Christoffel symbols associated with the Levi-Civita connection, and $X(T^{a_1\ldots a_p}_{b_1\ldots b_q})$ denotes the directional derivative of the component functions in the direction of X. Here $T^{a_1\ldots a_p}_{b_1\ldots b_q}$ is a function, so $X(T^{a_1\ldots a_p}_{b_1\ldots b_q}) = dT^{a_1\ldots a_p}_{b_1\ldots b_q}(X) = \sum_l X^l \partial_l T^{a_1\ldots a_p}_{b_1\ldots b_q}$

Example 18.1 (Levi-Civita Connection Acting on a Vector Field)**.**

- For a vector field V, which is a $(1, 0)$-tensor, the Levi-Civita connection acts as:

  $$\nabla_X V = X(V^a) + V^b\,\Gamma^a_b,$$

  where V^a denotes the components of V. Expanding this expression, we get:

  $$(\nabla_X V)^a = X(V^a) + V^b\,\Gamma^a_{b,k}X^k.$$

  In coordinates, if $X = X^i\partial_i$ and $V = V^j\partial_j$, this simplifies to:

  $$(\nabla_i V)^j = \partial_i V^j + \Gamma^j_{ik}V^k.$$

- For a 1-form α, which is a $(0, 1)$-tensor, the Levi-Civita connection acts by:

  $$\nabla_X \alpha = X(\alpha_b) - \alpha_a\,\Gamma^a_{b,k}X^k.$$

  Expanding in coordinates, we have:

  $$(\nabla_i \alpha)_j = \partial_i \alpha_j - \Gamma^k_{ij}\alpha_k.$$

18.1.2 General Properties of the Levi-Civita Connection on (p, q)-Tensors

The Levi-Civita connection satisfies the following properties:

- **Linearity**: $\nabla_X(T + S) = \nabla_X T + \nabla_X S$ for any (p, q)-tensors T and S.
- **Leibniz rule**: for any tensor product $T \otimes S$,

  $$\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes (\nabla_X S).$$

18.2 Lie Derivatives, Covariant Derivatives, and Exterior Differential on 1-Forms

In differential geometry, several notions of derivatives coexist and have different meanings. Here, we discuss the Lie derivative, covariant derivative, and exterior differential on 1-forms, there are more general formulas on k-forms.

18.2.1 Lie Derivative of 1-Forms

The Lie derivative measures the change of a tensor field along the flow generated by a vector field. For a smooth manifold M with a vector field X and a 1-form α, the Lie derivative of α along X, denoted $\mathcal{L}_X\alpha$, is defined by how α changes along the direction of X.

The Lie derivative of a 1-form α along X is given by:

$$(\mathcal{L}_X\alpha)(Y) = X \cdot (\alpha(Y)) - \alpha([X, Y]),$$

for any vector field Y, where $[X, Y]$ denotes the Lie bracket of X and Y. This definition emphasizes that $\mathcal{L}_X\alpha$ depends only on the directional derivative of α along X and the commutator structure of X and Y.

More generally, the Lie derivative of a k-form ω is obtained iteratively from the following properties:

- **Leibniz rule:** $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X\beta)$, and
- **commutation:** $\mathcal{L}_X d\alpha = d\mathcal{L}_X\alpha$.

18.2.2 Covariant Derivative of 1-Forms

For a 1-form α and vector fields X and Y on M, the covariant derivative of α in the direction of X is written $\nabla_X\alpha$ and defined as:

$$(\nabla_X\alpha)(Y) = X \cdot (\alpha(Y)) - \alpha(\nabla_X Y),$$

where $\nabla_X Y$ is the covariant derivative of Y along X. This derivative measures the rate of change of α in the direction of X, accounting for how the manifold's connection structure affects the relationship between X and Y.

18.2.3 Exterior Differential of 1-Forms

The exterior differential d provides a way to generalize the concept of the gradient to differential forms. For a 1-form α, the exterior differential $d\alpha$ is a 2-form defined by:

$$(d\alpha)(X, Y) = X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X, Y]),$$

where X and Y are vector fields on M. This formula shows that $d\alpha$ measures the infinitesimal change of α in the plane spanned by X and Y. The exterior differential is central in differential forms calculus and leads to concepts such as closed and exact forms. Note in particular the very useful formula:

$$d\alpha(X, Y) = (\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X),$$

saying that $d\alpha$ is essentially the antisymmetric part of $\nabla\alpha$.

Similarly, the exterior derivative can be computed from Lie derivatives. On 1-forms, this formula reduces to

$$d\alpha(X, Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X, Y]).$$

18.2.4 Cartan's Magic Formula

For a 1-form α and a vector field X, **Cartan's "magic" formula** states:

$$\mathcal{L}_X \alpha = d(i_X \alpha) + i_X(d\alpha),$$

where $i_X \alpha$ denotes the interior product (or contraction) of X with α. This elegant formula, that also holds on k-forms is the main tool to compute the Lie derivatives of differential forms.

Proof. To prove Cartan's magic formula, we compute $\mathcal{L}_X \alpha$ directly using its definition and the properties of the exterior differential and contraction:

$$(\mathcal{L}_X \alpha)(Y) = X \cdot (\alpha(Y)) - \alpha([X, Y]).$$

By definition, $d(i_X \alpha)(Y) = Y \cdot (\alpha(X)) - \alpha([Y, X])$, and $i_X(d\alpha)(Y) = d\alpha(X, Y)$. Combining these, we obtain the formula:

$$\mathcal{L}_X \alpha = d(i_X \alpha) + i_X(d\alpha).$$

□

18.3 The rough Laplacian on tensors

Intuition 18.2. The Levi-Civita connection $\nabla : \Omega^k(M) \to \Omega^1 \otimes \Omega^k$ on tensors also has a formal adjoint denoted $\nabla^* : \Omega^1 \otimes \Omega^k(M) \to \Omega^k$. It lets one define another, co-called *rough* Laplacian $\nabla^*\nabla$. Like the Hodge Laplacian on differential forms, it generalizes the Laplace-Beltrami operator on functions.

For a 1-form $\alpha = \alpha_i \, dx^i$, the Levi-Civita connection $\nabla\alpha \in \Omega^1 \otimes \Omega^1$ is a tensor with components given by:

$$(\nabla\alpha)_{ij} = \partial_i \alpha_j - \Gamma^k_{ij} \alpha_k,$$

where Γ^k_{ij} are the Christoffel symbols of the metric connection, representing the Levi-Civita connection in coordinates.

The **adjoint** ∇^* of ∇ is an operator that "traces out" over the derivative in a way similar to a divergence. Specifically, it reduces the rank of the $\nabla\alpha$ tensor by one.

Definition 18.3. On a 1-form α, the operator ∇^* is given by:

$$(\nabla^* \alpha) = -g^{ij} \nabla_i \alpha_j,$$

where g^{ij} are the components of the inverse metric tensor. On a 2-tensor α, or equivalently on $\Omega^1 \otimes \Omega^1$,

$$(\nabla^* \alpha)_k = -g^{ij} \nabla_i \alpha_{jk}.$$

Thus, ∇^* takes the trace over the covariant derivative, reducing the degree of the form by one.

It satisfies, for $\alpha \in \Omega^1(M)$ a 1-form and $\beta \in \Omega^1(M) \otimes \Omega^1(M)$ a 2-tensor

$$\langle \nabla\alpha, \beta \rangle = \langle \alpha, \nabla^* \beta \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product over the manifold with respect to the Riemannian volume form vol_g.

Let (E, ∇) be a bundle equipped with a connection over an oriented Riemannian manifold (M^n, g). For instance $(E, \nabla) = (\Omega^1(M), \nabla)$ is equipped with the induced Levi-Civita connection.

Definition 18.4. We define the **rough Laplacian**, or **Bochner Laplacian**, to be the operator $\nabla^*\nabla$ acting on sections of E.

Example 18.5.

- For a smooth function f, in coordinates, this becomes:

$$\Delta f = -g^{ij}\nabla_i\nabla_j f = -g^{ij}\left(\partial_i\partial_j f - \Gamma_{ij}^k\partial_k f\right) = -\frac{1}{\sqrt{\det(g_{ij})_{ij}}}\,\partial_i\left(\sqrt{\det(g_{ij})_{ij}}\,g^{ij}\,\partial_j f\right)$$

- The Rough Laplacian (or Bochner Laplacian) applied to a 1-form $\alpha = \alpha_i dx^i$ is defined in local coordinates, as:

$$\Delta\alpha_k = g^{ij}\left(\partial_i\partial_j\alpha_k - \Gamma_{ij}^m\partial_m\alpha_k - (\partial_i\Gamma_{jk}^m)\alpha_m - \Gamma_{jk}^m\partial_i\alpha_m + \Gamma_{ij}^m\Gamma_{mk}^n\alpha_n + \Gamma_{ik}^m\Gamma_{jm}^n\alpha_n\right)$$

Note 18.6. On functions, the rough Laplacian also coincide with the Laplace-Beltrami operator.

The rough Laplacian $\nabla^*\nabla$ is thus a second-order differential operator acting on the components of α. Thus, $\nabla^*\nabla$ serves as **another** natural generalization of the Laplace operator for differential forms, acting as a second-order operator that generalizes the idea of a Laplacian to curved spaces and arbitrary tensor fields.

18.4 Bochner Technique and cohomology

Intuition 18.7. In differential geometry, many, many Laplacian-like operators (i.e. becoming the usual Laplacian on coordinate functions of tensors on Euclidean space) coexist. They turn out to all be equal up to a *zeroth* order "differential" operator involving the curvature of the underlying manifold (or fiber bundle). Formulae relating different Laplacians are often called **Weitzenböck formulae**.

Using the Levi-Civita connection, we obtain a Laplacian $\nabla^*\nabla$ acting on 1-forms, which differs from the Hodge-de Rham Laplacian due to the following formula.

Lemma 18.8 (Bochner Formula)**.** Let (M^n, g) be an oriented Riemannian manifold. Then, for any 1-form α on M, we have

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha),$$

where $\mathrm{Ric}(\alpha)$ represents the Ricci curvature term.

Remark 18.9. There is a similar formula, known as the **Weitzenböck formula**, for k-forms. The difference $\Delta\alpha - \nabla^*\nabla\alpha$ is a zero-th order term involving the curvature of M.

Proof. Let X be a vector fields and $(e^i)_i$ be a g-orthonormal basis of T_pM. Assuming that we are in normal coordinates, we can choose such a family so that $\nabla e^i = 0$ at p and $\nabla X = 0$, as well as $[X, e^i] = 0$.

We recall that for any 1-form α, the exterior derivative $d\alpha$ satisfies

$$d\alpha(X, e^i) = (\nabla_X\alpha)(e^i) - (\nabla_{e^i}\alpha)(X).$$

Thus, we have

$$d^*d\alpha(X) = -\sum_{i=1}^n(\nabla_{e^i}d\alpha)(e^i, X) = -\sum_{i=1}^n(\nabla_{e^i}\nabla_{e^i}\alpha)(X) + (\nabla_{e^i}\nabla_X\alpha)(e^i),$$

where the last equality holds at a point p, with the chosen fields (e^i) and X parallel at p.

Similarly, we have $d^*\alpha = -\sum_{i=1}^n(\nabla_{e^i}\alpha)(e^i)$, giving

$$dd^*\alpha(X) = -\sum_{i=1}^n\nabla_X(\nabla_{e^i}\alpha)(e^i) = -\sum_{i=1}^n(\nabla_X\nabla_{e^i}\alpha)(e^i).$$

Therefore, at point p, comparing with the previous formula, we obtain

$$\Delta\alpha = \nabla^*\nabla\alpha + \sum_{i=1}^{n}(R(e_i,\cdot)\alpha)(e_i) = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha).$$

Note that here, we abusively identify: $(R(e_i,\cdot)\alpha)(e_i) = (R(e_i,\cdot)\alpha^\#)(e_i) \in \Omega^1(M)$ and $\mathrm{Ric}(\alpha) = \mathrm{Ric}(\alpha^\#,\cdot) \in \Omega^1(M)$. □

Corollary 18.10. If (M^n, g) is a compact connected oriented Riemannian manifold, then:

- If $\mathrm{Ric} > 0$, then $b_1(M) = 0$.
- If $\mathrm{Ric} \geq 0$, then $b_1(M) \leq n$, with equality if and only if (M, g) is a flat torus.

Proof. Suppose now that M is compact. By Hodge theory, an element of $H^1(M)$ is represented by a harmonic 1-form α. Applying the Bochner formula, we get

$$\nabla^*\nabla\alpha + \mathrm{Ric}(\alpha) = 0.$$

Taking the inner product with α, we obtain

$$\|\nabla\alpha\|^2 + (\mathrm{Ric}(\alpha), \alpha) = 0.$$

If $\mathrm{Ric} > 0$, this equation implies $\nabla\alpha = 0$ and $\mathrm{Ric}(\alpha) = 0$. When $\mathrm{Ric} > 0$, we have $\alpha = 0$; if $\mathrm{Ric} \geq 0$, we conclude that α is parallel.

Conclusion: the cohomology is represented by parallel forms. But if M is connected, a parallel form is determined by its values at a single point. Thus, $\dim H^1 \leq n$, with equality only if M has a basis of parallel 1-forms. This implies that M is flat. □

Remark 18.11. This is a typical example of using a Weitzenböck formula to derive vanishing theorems for cohomology.

Note 18.12. Beautiful Weitzenböck formulae related to other topologically or physically relevant equations can be derived.

- On a manifold with "positive isotropic curvature", there are no harmonic 2-forms.
- On a manifold with positive scalar curvature, there are no harmonic spinors: the index of the Dirac operator, which is a topological invariant denoted $\hat{A}$ vanishes on manifolds admitting a metric with positive scalar curvature. This is because of Lichnerowicz formula:

$$D^2 = \nabla^*\nabla + \frac{1}{4}S.$$

- on a manifold with positive scalar curvature, there are no solutions to Seiberg-Witten equations: the Seiberg-Witten invariant SW vanishes on manifolds admitting a metric with positive scalar curvature because of a "twisted" version of the above Lichnerowicz formula.

18.5 Bochner technique and symmetries

Theorem 18.13. Let (M^n, g) be a compact manifold with negative Ricci curvature, then it cannot admit a Killing vector field.

Proof. Define $B_g : \mathrm{Sym}^2(T^*M) \to \Omega^1(M)$ by $B_g h = \delta_g\big(h - \frac{1}{2}\operatorname{tr}_g h\big)$, where $(\delta h)_i = -g^{jk}\nabla_j h_{ki}$. We once again have a nice Weitzenböck formula:

$$B_g \mathcal{L}_X g = \nabla^*\nabla X - \mathrm{Ric}(X,\cdot)^\#.$$

If X is a Killing vector field, then $\mathcal{L}_X g$, hence $B_g \mathcal{L}_X g = 0$, which integrated by parts against X yields:

$$\begin{aligned} 0 &= \int_M \langle B_g \mathcal{L}_X g, X\rangle dv \\ &= \int_M \langle \nabla^*\nabla X - \mathrm{Ric}(X,\cdot)^\#, X\rangle dv \\ &= \int_M |\nabla X|^2 - \mathrm{Ric}(X,X) dv. \end{aligned}$$

If $\mathrm{Ric} < 0$, this is impossible unless $X \equiv 0$. □

18.6 Weitzenböck formulae and eigenvalue estimates

Intuition 18.14. When the curvature term in a Weitzenböck formula has a sign, then one can compare the spectra of two different differential operators. By integration by parts an operator of the form PP^* (e.g. $d\delta$, δd or $\nabla^*\nabla$) always has nonnegative spectrum.

18.6.1 Obata eigenvalue estimate

Theorem 18.15 (Obata's Eigenvalue Estimate)**.** Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 2$ with Ricci curvature satisfying

$$\mathrm{Ric}(X,X) = (n-1)\lambda g(X,X)$$

for some constant $\lambda > 0$ and for all tangent vectors X. Then the first non-zero eigenvalue λ_1 of the Laplacian on M satisfies

$$\lambda_1 \geq n\lambda,$$

Proof. Let ϕ be a smooth eigenfunction of the Laplacian on M, corresponding to the first non-zero eigenvalue λ_1. Then ϕ satisfies

$$\delta d\phi = \Delta\phi = \lambda_1\phi.$$

Now consider the 1-form $\alpha = d\phi$ which satisfies:

$$\begin{aligned} \Delta_H \alpha &= (\delta d + d\delta)d\phi \\ &= (d\delta)d\phi \\ &= d(\Delta\phi) \\ &= \lambda_1 d\phi \\ &= \lambda_1\alpha. \end{aligned}$$

Consider the integration by parts of the proof of Bochner's formula: it yields

$$\|\nabla\alpha\|_{L^2}^2 + \int_M \mathrm{Ric}(\alpha^\#, \alpha^\#) dv = \lambda_1 \|\alpha\|_{L^2}^2.$$

Now, $\mathrm{tr}(\nabla\alpha) = tr(\nabla d\phi) = \Delta\phi$, so

$$\|\nabla\alpha\|_{L^2}^2 = \left\|\nabla\alpha - \frac{1}{n}(\Delta\phi)g\right\|_{L^2}^2 + \left\|\frac{1}{n}(\Delta\phi)g\right\|_{L^2}^2 \geqslant \frac{1}{n}\|\Delta\phi\|_{L^2}^2,$$

since pointwise $|g|^2 = n$.

We therefore find $\lambda_1 \geqslant n\lambda$ as stated.

$\square$

Note 18.16. In the equality case, $\nabla d\phi = \frac{1}{n}(\Delta\phi)g = \lambda\phi g$, and one can show that this implies that the metric is the round sphere of radius $\frac{1}{\sqrt{\lambda}}$ of constant sectional curvature λ.

19 The space of Einstein metrics

In differential geometry, algebraic geometry, topology and physics, one is often interested in the set of solutions to an equation modulo its obvious invariances. The resulting set is called a **moduli space**. In this last section of the notes, we focus on the Einstein equation and the associated moduli space of Einstein metrics.

19.1 Einstein Equation

An Einstein metric is a metric with constant Ricci curvature. More precisely, a metric g is called *Einstein* if there exists a constant Λ, referred to as the *Einstein constant*, such that:

$$\mathrm{Ric}(g) = \Lambda g. \tag{1}$$

This equation originates from physics (in the Lorentzian context), as it characterizes the critical points of the *Hilbert-Einstein action* under a fixed volume:

$$\overline{S}(g) := \int_M S(g)\,\mathrm{d}v_g.$$

This functional is a natural quantity that measures the spatial average of the scalar curvature, itself being the directional average of the Riemannian curvature.

Another motivation, more geometric in nature, for studying this equation is that it imposes a curvature to be constant. Enforcing the sectional curvatures to be constant only permits three local geometries and imposes overly restrictive topological conditions, except in dimension 2. This makes it an excessively rigid condition. Conversely, requiring the scalar curvature to be constant is always possible but results in infinite-dimensional solution spaces, except in dimension 2, making it too flexible. The Einstein condition strikes a balance, providing as many equations as there are degrees of freedom for the metric (the Ricci curvature and the metric are symmetric 2-tensors at each point), and its solutions are heuristically neither too flexible nor too rigid.

Note 19.1. There are many more reasons to consider Einstein metrics in dimension 4. They minimize the L^2-norm of curvature and many natural quantities, and are the selfdual metrics of the theory. They are often referred to as *gravitational instantons* in the physics literature by analogy with the Yang-Mills setting.

19.1.1 Invariances of the Einstein Equation

The Einstein equation is invariant under scaling because for any metric g and any positive constant s^2, the Ricci curvature tensor satisfies:

$$\mathrm{Ric}(s^2 g) = \mathrm{Ric}(g).$$

Thus, if g satisfies $\mathrm{Ric}(g) = \Lambda g$, then:

$$\mathrm{Ric}(s^2 g) = \Lambda g = \frac{\Lambda}{s^2}(s^2 g).$$

Similarly, it is invariant under the pullback action of diffeomorphisms. For any diffeomorphism $\phi : M \to M$, we have:

$$\mathrm{Ric}(\phi^* g) = \phi^* \mathrm{Ric}(g),$$

and hence, if $\mathrm{Ric}(g) = \Lambda g$, then $\mathrm{Ric}(\phi^* g) = \Lambda(\phi^* g)$. This implies that whenever an Einstein metric exists on a manifold, there is an infinite-dimensional space of such metrics. In particular, the Einstein equation, as stated, is **not elliptic**.

19.1.2 The Einstein Equation in Bianchi Gauge

The Bianchi gauge is a natural choice for rendering the Einstein equation elliptic, leveraging the Bianchi identity.

Definition 19.1 (Bianchi Operator and Gauge)**.** The **Bianchi operator** applied to a symmetric 2-tensor h is defined as:

$$B_g(h) := \delta_g \left(h - \frac{\mathrm{tr}_g h}{2} g \right),$$

where δ_g is the divergence operator. We say that h is in **Bianchi gauge** with respect to g if $B_g h = 0$.

Theorem 19.2 (Slice theorem, Bianchi version)**.** Let (M, g_0) be a Riemannian metric with $\mathrm{Ric}(g_0) < 0$. Then, there exists a neighborhood $\mathcal{U}$ of g_0 in the space of $(C^{k,\alpha})$ metrics so that for any $g \in \mathcal{U}$, there exists a unique diffeomorphism ϕ close to the identity so that:

$$B_{g_0}(\phi^* g) = 0,$$

i.e., any metric can be put in Bianchi gauge with respect to g_0 up to the action of the group of diffeomorphisms.

The space of metrics satisfying $B_{g_0}(g) = 0$ is the **slice** and is transverse to the orbits of the diffeomorphism group.

Sketch of proof. This is an application of the implicit function theorem between Banach spaces. Indeed, we are searching for zeros of the map

$$\Psi(g, \phi) = B_{g_0}(\phi^* g)$$

close to the solution (g_0, Id). The tangent space to the diffeomorphism group at the identity is identified with vector fields (or 1-forms), and the linearization of $\phi \mapsto \Psi(g_0, \phi)$ is $B_{g_0}\delta^*_{g_0}$ at the identity.

Using the identity $B_g \delta^*_g = \frac{1}{2}(\nabla^*_g \nabla_g - \mathrm{Ric}(g))$ and integrating by parts, we derive:

$$0 = \int_M |\delta(B_{g_0}(g))|^2 \, \mathrm{d}v_g - \int_M \langle \mathrm{Ric}(g)(B_{g_0}(g)), B_{g_0}(g) \rangle \, \mathrm{d}v_g.$$

□

Note 19.3. Ebin-Palais prove a similar slice theorem using the divergence-free gauge without requiring that the Ricci curvature is negative.

This gauge choice is justified by the following lemma, which clarifies the linearization of Ricci and scalar curvatures.

Lemma 19.2 (Linearization of Ricci and Scalar Curvatures)**.** The linearization of the Ricci curvature at a metric g in the direction of h is given by:

$$\mathrm{Ric}'_g(h) := \partial_t \mathrm{Ric}(g + th)_{|t=0} = \frac{1}{2}\left(\nabla^* \nabla h - 2\delta^* \delta h - \nabla^2 \mathrm{tr} h - 2\mathring{\mathcal{R}}(h) + \mathrm{Ric} \circ h + h \circ \mathrm{Ric} \right),$$

where $\mathring{\mathcal{R}}(h)$ is a curvature term defined by:

$$\mathring{\mathcal{R}}(h)(X, Y) := \sum_i h(R(e_i, X)Y, e_i),$$

and the composition $\circ$ is defined via the metric. The linearization of the scalar curvature is:

$$S'_g(h) := \partial_t S(g+th)_{|t=0} = -\Delta(\operatorname{tr} h) + \delta\delta h - \langle \operatorname{Ric}, h\rangle.$$

In particular, if g_0 is Einstein with $\operatorname{Ric}(g_0) = \Lambda g_0$, and h is in the Bianchi gauge satisfying $\int_M \operatorname{tr}_{g_0} h \, dv_{g_0} = 0$, the linearization of the operator $g \mapsto \operatorname{Ric}(g) - \frac{\overline{S}(g)}{n} g$ at g_0 is given by the elliptic operator:

$$P_{g_0}(h) := \frac{1}{2}\nabla^*\nabla h - \mathring{\mathcal{R}}_{g_0}(h).$$

The Einstein and Bianchi gauge equations can be combined into a single elliptic equation:

Lemma 19.3. Let (M, g_0) be a compact Einstein manifold with $\operatorname{Ric}(g_0) = \Lambda g_0$ and $\Lambda < 0$. Suppose g is another metric sufficiently close to g_0 in the C^2-norm. Then:

$$\operatorname{Ric}(g) = \frac{\overline{S}(g)}{n} g, \quad \text{and} \quad B_{g_0}(g) = 0,$$

if and only if:

$$\Phi_{g_0}(g) := \operatorname{Ric}(g) - \frac{\overline{S}(g)}{n} g + \delta_g^* B_{g_0}(g) = 0,$$

where δ_g^* is the formal adjoint of the divergence operator, given by $\delta_g^*(X) = \frac{1}{2}\mathcal{L}_X g$, with $\mathcal{L}_X$ the Lie derivative.

Proof. The first implication is straightforward. For the reverse direction, suppose:

$$\operatorname{Ric}(g) - \frac{\overline{S}(g)}{n} g + \delta_g^* B_{g_0}(g) = 0.$$

Applying the Bianchi operator B_g, which satisfies $B_g \operatorname{Ric}(g) = 0$, we obtain:

$$B_g(\delta_g^* B_{g_0}(g)) = 0.$$

Using the identity $B_g \delta_g^* = \frac{1}{2}(\nabla_g^* \nabla_g - \operatorname{Ric}(g))$ and integrating by parts, we derive:

$$0 = \int_M |\delta(B_{g_0}(g))|^2 \, dv_g - \int_M \langle \operatorname{Ric}(g)(B_{g_0}(g)), B_{g_0}(g)\rangle \, dv_g.$$

If g is sufficiently close to g_0 in the C^2-norm, the largest eigenvalue of $-\operatorname{Ric}(g)$ is less than half the largest eigenvalue of $-\operatorname{Ric}(g_0)$. This implies $B_{g_0}(g) = 0$. $\square$

Remark 19.4. In cases where the manifold has infinite volume, $\overline{S}$ is ill-defined and is replaced by $n\Lambda$. This fixes the Einstein constant but prevents Ricci-flat metrics from deforming into non-flat Einstein metrics.

19.2 Moduli Space of Einstein Metrics

Let M be a compact differentiable manifold of dimension n. The set of smooth Riemannian metrics on M is denoted $\mathcal{M}(M)$, a subset of symmetric 2-tensors on M, $S^2(M)$. The subset of Einstein metrics on M is denoted $\mathcal{E}(M) \subset \mathcal{M}(M)$.

Since the Einstein condition is stable under scaling and the action of the group of smooth diffeomorphisms, we define the **moduli space of Einstein metrics** on M as the quotient:

$$\mathbf{E}(M) := \left\{ g \,\middle|\, \exists \Lambda \in \mathbb{R},\ \operatorname{Ric}(g) = \Lambda g,\ \operatorname{Vol}(M, g) = 1 \right\} / \mathcal{D}(M),$$

where $\mathcal{D}(M)$ is the group of diffeomorphisms acting by pullback on metrics.

19.2.1 Local Structure of the Moduli Space Near a Smooth Einstein Metric

To describe the local structure of $\mathbf{E}(M)$ near a smooth Einstein metric g_0 on M, we study a representative subset of equivalence classes. This subset, called a *slice*, consists of metrics that satisfy specific conditions near g_0. These conditions fix the scaling and diffeomorphism actions.

Definition 19.5 (Pre-moduli Space)**.** Let M be a differentiable manifold and $g_0 \in \mathbf{E}(M)$. The pre-moduli space near the metric g_0, denoted $\mathbb{E}_{g_0}(M)$, is the set of metrics g such that:

1. g is Einstein with the same Einstein constant as g_0:
$$\mathrm{Ric}(g) = \Lambda g,$$
which fixes the scale when $\Lambda \neq 0$;

2. g has the same volume as g_0:
$$\mathrm{Vol}(M, g) = \mathrm{Vol}(M, g_0),$$
which fixes the scale;

3. g is in Bianchi gauge with respect to g_0:
$$\delta_{g_0} g = 0,$$
which fixes the action of diffeomorphisms, up to isometries.

Remark 19.6. The first two conditions are consistent (and redundant) in the case $\Lambda \neq 0$ if we remain in the same connected component of the moduli space. Indeed, Einstein metrics are critical points of the Einstein-Hilbert functional:

$$\int_M S(g)\, \mathrm{d}v_g = n\mathrm{Vol}(M, g)\Lambda,$$

which is constant on each connected component of the moduli space. However, it is not immediately clear whether there can be accumulations of elements from different connected components of metrics near a given metric. Near a smooth metric, the answer is no, and the Einstein constant Λ is indeed fixed due to a deep theorem of Koiso, Theorem 19.4.

For this reason, we relax the first condition to:

$$E(g) := \mathrm{Ric}(g) = \frac{\overline{S}}{n} g,$$

where $\overline{S}(g)$ is the mean scalar curvature. This operator is compatible with the Bianchi gauge because it satisfies $B_g E(g) = 0$ for all metrics g.

Remark 19.7. To rigorously study slices of the actual moduli space $\mathbf{E}(M)$, we must also quotient by the action of the isometry group of g_0, which is finite-dimensional. The above local pre-moduli space covers the moduli space.

19.2.2 Analytic Structure of the Pre-Moduli Space

The pre-moduli space $\mathbb{E}_{g_0}(M)$ inherits a natural analytic structure. Specifically, near g_0, the Einstein metrics satisfying the divergence-free gauge condition and volume normalization are precisely the zero set of the operator:

$$\Phi_{g_0}(g) := E(g) + \delta_g^* B_{g_0} g.$$

Lemma 19.3, might be rewritten as follows.

Lemma 19.8. Let (M^n, g_0) be a compact Einstein manifold with $\mathrm{Vol}(M, g_0) = 1$. There exists $\varepsilon > 0$ such that:

$$\Phi_{g_0}^{-1}(\{0\}) \cap B_{C^{1,\alpha}}(\varepsilon) \cap \{g \mid \mathrm{Vol}(M, g) = 1\}$$

is exactly the set of Einstein metrics of volume 1 in Bianchi gauge with respect to g_0 in $B_{C^{1,\alpha}}(\varepsilon)$.

Theorem 19.4 (Koiso, Bianchi version)**.** Let (M, g_0) be an Einstein metric with $\mathrm{Ric} < 0$. Then, the local premoduli space of Einstein metrics close to g_0 coincides with

$$\Phi_{g_0}^{-1}(\{0\}) \cap B_{C^{1,\alpha}}(\varepsilon) \cap \{g \mid \mathrm{Vol}(M, g) = 1\}.$$

This set is **real analytic**.

Sketch of proof. This is an application of the implicit function theorem to the operator Φ_{g_0}. It however does not generally have an invertible linearization and one needs to perform a Lyapunov-Schmidt reduction in order to obtain a Kuranishi map.

More concretely, denote $\ker P_{g_0}$ is the kernel (and cokernel) of the self-adjoint operator P_{g_0}, the linearization of Φ_{g_0} at g_0. Then, for any $v \in \ker P_{g_0}$, the map

$$\Psi_v : (g = g_0 + v + h, w) \mapsto \Phi_{g_0}(g) - w$$

for $h \in (\ker P_{g_0})^{\perp}$ and $v \in \ker P_{g_0}$ is analytic in its arguments and has an invertible linearization in the direction (h, w) at $v = 0$. This is a form of the **Lyapunov-Schmidt reduction**.

Applying the suitable implicit function theorem for real-analytic maps between Banach spaces, one obtains the so-called **Kuranishi map**: for any $v \in \ker P_{g_0}$ (a finite-dimensional set), one obtains a real analytic map

$$v \mapsto (g_v, w_v)$$

so that $g_v - g_0 - v \perp \ker P_{g_0}$ and $\Phi_{g_0}(g_v) = -w_v$. The local set of gauged and unit-volume Einstein metrics is therefore exactly

$$\left\{g_v\,,\ \ v \in \ker P_{g_0},\ \text{small and so that } w_v = 0\right\}.$$

This is the zero-set of a real-analytic map and consequently has the structure of a real-analytic variety. ☐

This result has a lot of consequences for the local structure of the set of Einstein metrics on a given Riemannian manifold.

Corollary 19.5. The set of Einstein metrics on a given Riemannian manifold is locally connected, locally finite dimensional. There are only countably many possible Einstein constants on a given Einstein manifold. Connected components of the moduli space cannot accumulate near a smooth Einstein metric.

Note 19.6. Once one studies the **global** properties of the entire moduli space of Einstein metrics, these finiteness properties hold until dimension 3, but fail from dimension 5. A major open question is whether the set of Einstein metrics is manageable in its critical dimension 4.